A symbolic approach to computing with holonomic functions

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Automata and formal languages: mathematical and applicative aspects.

Outline

1. Holonomic functions
   - Basic definitions: the HOLO class
   - The membership problem for HOLO: known solutions
2. Symbolic approach
   - Canonical representation
   - D-closed sets and CF grammars
   - Differential equations for holonomic functions

Weyl Algebra

Definition ($A(\mathbf{Q})$)

The Weyl Algebra $A(\mathbf{Q})$ is the ring of linear differential operators in the variable x with polynomial coefficients.

- Operators:
  Multiplication by x: $X(f(x)) = xf(x)$.
  Derivation by x: $D(f(x)) = f'(x)$.
- Pseudo-commutation rule: $DX = XD + 1$.

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Holonomic functions

Definition (annihilator)

$w \in A(\mathbf{Q})$ is an annihilator for $f(x)$ if $w(f(x)) = 0$.

Definition (holonomic function)

A function $f(x)$ is holonomic if it admits an annihilator $w \in A(\mathbf{Q})$.

Example

The function $f(x) = \frac{e^x \sin(x)}{1-x}$ is holonomic, since

$$((X-1)D^2 + (4-2X)D - 4 + 2X)(f(x)) = 0.$$

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Membership Problem for HOLO

HOLO is the class of the holonomic functions.

Problem (Membership for HOLO)

Input: An analytic function $f(x)$.

Output: 1 if $f(x)$ is holonomic, 0 otherwise.

A constructive solution: Find an annihilator for $f(x)$.

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Membership Problem for HOLO (cont.)

ALGEBRAIC APPROACH: It exploits

- closure properties of HOLO w.r.t. $+, \cdot, \odot, \ldots$;
- RAT $\subset$ ALG $\subset$ HOLO;

Limitations: Not all holonomic functions are obtained by means of the closure properties.

ANALYTIC APPROACH: It exploits

- Holonomic functions have finitely many singularities;
- well known asymptotic form of coefficients near regular singularities.

Limitations: Useful for proving non-holonomicity, it does not provide annihilators.

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Symbolic approach

The framework:

- a finite set $\mathcal{F}$ of functions having a derivation rule;
- CLOSE($\mathcal{F}$), i.e. the closure of $\mathcal{F}$ with respect to sum, product and composition;
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Canonical representation

Canonical representation of $h(x)$:

$$h(x) = \sum_{j=0}^{k} r_j(x)\, a_j(x) \qquad \text{where}$$

- $r_j(x)$ are rational functions;
- $a_0(x) = 1$;
- $a_j(x)$, $1 \leq j \leq k$ are finite products of nonrational functions,

$$a_j(x) = \prod_{l=1}^{e_j} t_{jl}(x), \qquad t_{jl} \in \text{CLOSE}(\mathcal{F})$$

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Example

$$h(x) = x^2 + \frac{1}{1-x}\, e^{\arcsin(x)} \ln(1-x) =$$
$$= r_0(x)\, a_0(x) + r_1(x)\, a_1(x) =$$
$$= r_0(x)\, a_0(x) + r_1(x)\, t_{10}(x)\, t_{11}(x)$$

- $r_0(x) = x^2$;
- $a_0(x) = 1$;
- $r_1(x) = \frac{1}{1-x}$;
- $a_1(x) = t_{10}(x) t_{11}(x)$;
- $t_{10}(x) = e^{\arcsin(x)}$;
- $t_{11}(x) = \ln(1-x)$.

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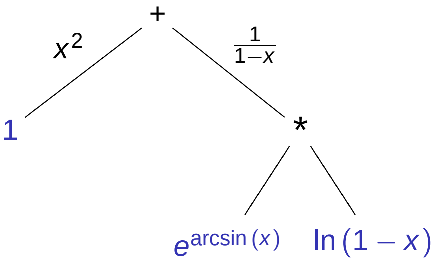

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D-close sets

Definition (D-closed set)

A set $\mathcal{F}$ of functions is said D-closed if the derivative of any function in $\mathcal{F}$ can be expressed as a finite sum (with rational coefficients) of products of elements in $\mathcal{F}$.

$\mathcal{F}$ D-closed $\implies \forall t(x) \in \mathcal{F}$, $t'(x)$ has a canonical rep. in $\mathcal{F}$.

Example

$\mathcal{F} = \{\sin(\cos(x)), \cos(\cos(x)), \cos(x), \sin(x)\}$ is D-closed,

- $D(\sin(\cos(x))) = -\sin(x)\cos(\cos(x))$;
- $D(\cos(\cos(x))) = \sin(x)\sin(\cos(x))$;
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Definition ($\sigma(f)$)

$\sigma(f)$ is the smallest integer k such that

$$\exists A \text{ D-closed}, \ \operatorname{card}(A) = k, \ f \in A$$

Lemma

Let $\mathcal{F}$ be a finite set of functions and $f(x) \in CLOSE(\mathcal{F})$. Then,

$$\forall g \in \mathcal{F}, \ \sigma(g) < \infty \implies \sigma(f) < \infty$$

As a consequence, given a canonical representation

$$h(x) = \sum_{i=0}^{k} r_i(x) \prod_{j=1}^{e_i} t_{ij}^{m_{ij}}(x),$$

there is a finite D-closed set $\mathcal{B}(h)$ containing $\{t_{ij}\}$.

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A grammar for finite D-closed sets

Let $\mathcal{B}(h) = \{t_1(x), \ldots, t_q(x)\}$ be a finite D-closed set:

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\begin{aligned}
D(t_1(x)) &= \sum_{i=1}^{k_1} r_{1i}(x) \prod_{j=1}^{q} t_j^{m_{1ij}}(x) \\
&\vdots \\
D(t_q(x)) &= \sum_{i=1}^{k_q} r_{qi}(x) \prod_{j=1}^{q} t_j^{m_{qij}}(x)
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A grammar for finite D-closed sets (continued)

$G_{\mathcal{B}(h)} = \langle V, \Sigma, P, S \rangle$, where

- $V = \{T_1, \ldots, T_q, S\}$,
- $\Sigma = \{t_1, \ldots, t_q\}$,
- P is the set of productions

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\begin{aligned}
S &\rightarrow T_1 \mid T_2 \mid \ldots \mid T_q, \\
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Example ($h(x) = \sin(\cos(x))$)

$$\mathcal{B}(h) = \{\sin(\cos(x)), \sin(x), \cos(x), \cos(\cos(x))\}.$$

This leads to the grammar

$$G_{\mathcal{B}(h)} = \langle \{T_1, T_2, T_3, T_4, S\}, \{t_1, t_2, t_3, t_4\}, P, S\rangle$$

where
$$\begin{aligned} P = \{ & S \to T_1|T_2|T_3|T_4, \\ & T_1 \to t_1|T_3T_2, \quad T_2 \to t_2|T_3T_1, \\ & T_3 \to t_3|T_4, \quad T_4 \to t_4|T_3\} \end{aligned}$$

with $t_1 \equiv \sin(\cos(x))$, $t_2 \equiv \cos(\cos(x))$, $t_3 \equiv \sin(x)$, $t_4 \equiv \cos(x)$.

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$G_{\mathcal{B}(h)}$ and HOLO

Theorem

If the language $L(G_{\mathcal{B}(h)})$ is finite, then $h(x)$ is holonomic.

Proof (outline): Let ρ_c be the congruence generated by $t_i t_j = t_j t_i$ and observe that:

- the commutative image $L(G_{\mathcal{B}(h)})/_{\rho_c}$ is finite;
- $L(G_{\mathcal{B}(h)})/_{\rho_c}$ is a finite set of generators for $\{D^i(h(x))\}$.

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Finding an annihilator

Let $h(x)$ with $L(G_{\mathcal{B}(h)})$ finite. Then, for all $i \in \mathbb{N}$:

$$D^i(h(x)) = \sum_{j=1}^{k_i} r_{ij}(x) a_{ij}(x),$$

where $a_{ij}(x) = \prod_{l=1}^{e} t_{ij}^{m_{ij}}(x)$ ($m_{ij} \in \mathbb{N}$) and $t_{ij}(x) \in \mathcal{B}(h)$.

Note that:

- $L(G_{\mathcal{B}(h)})$ finite $\Longrightarrow \{a_{ij}(x)\}_{i,j>0}$ finite.
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Example (A system for $h(x) = e^{\arcsin(x)}$)

$$\mathcal{B}(h) = \{e^{\arcsin(x)}, \sqrt{1-x^2}\}$$

$$\begin{aligned} D^0(h(x)) &= a_{11}(x), \\ D^1(h(x)) &= \frac{1}{1-x^2} a_{22}(x), \\ D^2(h(x)) &= \frac{1-x^2}{x^4-2x^2+1} a_{31}(x) + \frac{x}{x^4-2x^2+1} a_{32}(x), \end{aligned}$$

where

- $a_{11}(x) = a_{31}(x) = e^{\arcsin(x)}$;
- $a_{22}(x) = a_{32}(x) = \sqrt{1-x^2} e^{\arcsin(x)}$.

Finding an annihilator (cont. I)

Let $(a_1(x), \ldots, a_n(x)) = \{a_{ij}(x)\}_{i,j>0}$. Then, the system

$$D^i(h(x)) = \sum_{j=1}^{n} r_{ij}(x) a_j(x), \qquad 0 \leq i \leq n,$$

can be written as

$$\mathbf{R} \cdot \mathbf{a} = \mathbf{v},$$

where

$$\begin{aligned} \mathbf{R} &= [r_{ij}(x)]_{(n+1)\times n}, \\ \mathbf{a} &= (a_1(x), \ldots, a_n(x))^T, \\ \mathbf{v} &= (D^0, \ldots, D^n)^T. \end{aligned}$$

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An annihilator $w \in A(\mathbb{Q})$ for $h(x)$ can be obtained by computing

$$w = \det(\mathbf{v}|\mathbf{R}),$$

where $\mathbf{v}|\mathbf{R}$ is the augmented matrix of the system

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If $\det(\mathbf{v}|\mathbf{R}) = 0$, then the matrix is singular. We compute the determinant of a square submatrix of the reduced echelon form of $\mathbf{v}|\mathbf{R}$.

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Example (A differential equation for $h(x) = e^{\arcsin(x)}$)

In the case of

$$h(x) = e^{\arcsin(x)}$$

$$\mathbf{v}|\mathbf{R} = \begin{bmatrix} D^0 & 1 & 0 \\ D^1 & 0 & 1 \\ D^2 & \frac{1-x^2}{x^4-2x^2+1} & \frac{x}{x^4-2x^2+1} \end{bmatrix}$$

$$\det(\mathbf{v}|\mathbf{R}) = (x^2-1)D^2 + xD + 1$$

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The symbolic approach

- allows to recognize a wide class of holonomic functions;
- provides annihilators for holonomic functions;
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Further developments and open problems

- Extension to multivariate functions and D-finite series.
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A non-holonomic function: $h(x) = \sin(\cos(x))$

$$\mathcal{B}(h) = \{\sin(\cos(x)), \sin(x), \cos(x), \cos(\cos(x))\}.$$

$G_{\mathcal{B}(h)} = \langle \{T_1, T_2, T_3, T_4, S\}, \{t_1, t_2, t_3, t_4\}, P, S \rangle$ where

$$P = \{S \to T_1|T_2|T_3|T_4, \quad T_1 \to t_1|T_3T_2, \quad T_2 \to t_2|T_3T_1, \quad T_3 \to t_3|T_4, \quad T_4 \to t_4|T_3\}.$$

- $\{t_3^{2k} t_1 | k \geq 0\} \subseteq L(G_{\mathcal{B}(h)}) \implies L(G_{\mathcal{B}(h)})$ not finite.
- $Span(\sin(x)^{2k} \sin(\cos(x))) \subseteq Span(D^i \sin(\cos(x)))$.

$h(x) = \sin(\cos(x))$ is not holonomic.

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