On the complexity of unary tiling-recognizable languages

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$$\begin{array}{ccc} \sharp & \sharp\cdots\sharp & \sharp \\ \sharp & & \sharp \\ \vdots & p & \vdots \\ \sharp & & \sharp \\ \sharp & \sharp\cdots\sharp & \sharp \end{array} = \hat{p}$$

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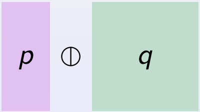

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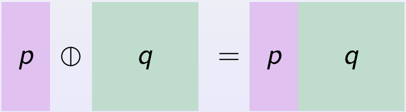

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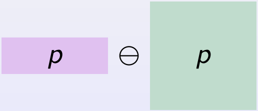

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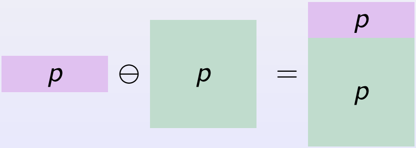

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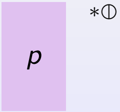

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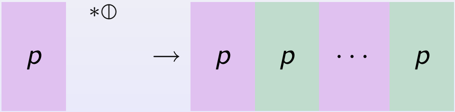

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$$\boxed{p}^{*\ominus}$$

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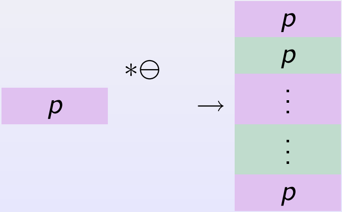

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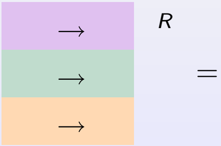

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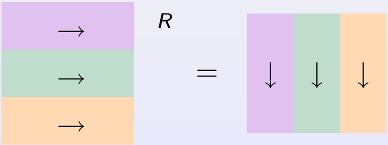

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- REC is a class of two-dimensional languages
- REC tries to extends the concept of regular string language
- REC can be defined using different approach:
  - regular expressions
  - online tessellation automata
  - logic formulas
  - tiling systems

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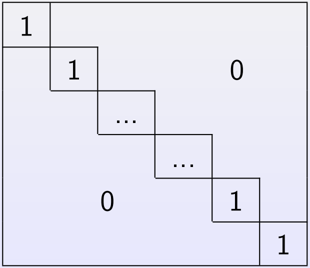
- A language L is in $\textsc{Rec}$ if there exists a tiling system such that $L = \pi(\mathcal{L}(\Theta))$.
- $\textsc{Rec}$ is closed w.r.t the operations $\cup, \varobar, \ominus, {}^{*\varobar}, {}^{*\ominus}, {}^{R}$.

Example: squares

- The set of squares with 1 on the main diagonal and 0 in all other position is a local language



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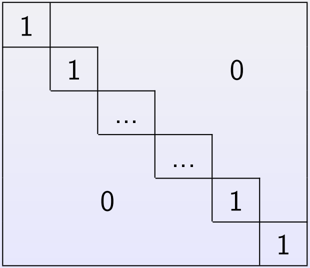

$$p = \begin{array}{|c|c|c|c|c|c|} \hline \sharp & \sharp & \sharp & \sharp & \sharp & \sharp \\ \hline \sharp & 1 & 0 & 0 & 0 & \sharp \\ \hline \sharp & 0 & 1 & 0 & 0 & \sharp \\ \hline \sharp & 0 & 0 & 1 & 0 & \sharp \\ \hline \sharp & 0 & 0 & 0 & 1 & \sharp \\ \hline \sharp & \sharp & \sharp & \sharp & \sharp & \sharp \\ \hline \end{array}$$

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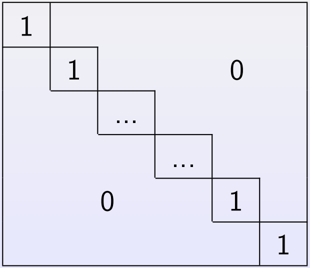
- The set of unary squares is in $\textsc{Rec}$ (it is the projection of the previous languages).



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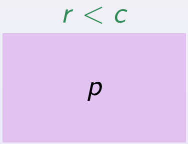

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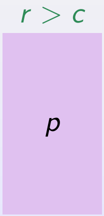

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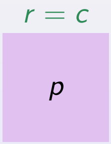

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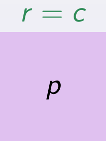

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Size of quasi-unary strings

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Ex: p

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such that, on every input x,

- M works within $|x|$ space,
- M executes ${}_{\circ}|x|$ head reversals at most.

Characterization of $\textsc{Rec}_1$

Theorem

Given any two-dimensional unary language L, the following statements are equivalents

- *L is in* $\textsc{Rec}_1$
- $\phi(L)$ *belongs to* $\textsc{N-LinSpaceRev}_{QU}$.

Recognizability implies the complexity bound

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  $\Longrightarrow \quad \phi(L)$ is in $\textsc{N-LinSpaceRev}_{QU}$.

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  - otherwise the generation has to be done column by column.
- The input is accepted if and only if such a generating process can be accomplished.

Tape

$\circ$	$\cdots$	$\circ$	h	$\circ$	$\cdots$	$\circ$

Tape

$\sharp$ p_{11}	$\cdots$	$\circ$	h	$\circ$	$\cdots$	$\circ$

$$\begin{array}{|c|c|}\hline \sharp & \sharp \\ \hline \sharp & p_{11} \\ \hline \end{array} \in \Theta$$

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$\sharp$ p_{11}	$\sharp$ $\cdots$	$\sharp$ p_{1m}	h	$\circ$	$\cdots$	$\circ$

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$\sharp$ p_{11}	$\sharp$ $\cdots$	$\sharp$ p_{1m}	$\sharp$ p_{1m+1}	$\circ$	$\cdots$	$\circ$

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$\sharp$	$\sharp$
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$\in \Theta$

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p_{11}	$\cdots$	p_{1m}	p_{1m+1}	p_{1m+2}	$\cdots$	p_{1n}

$\sharp$	$\sharp$	$\in \Theta$
$\cdots$	p_{1n}	

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Tape

$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$	p_{1n}
p_{11}	$\cdots$	p_{1m}	p_{1m+1}	p_{1m+2}	$\cdots$	p_{2n}

$$\begin{array}{|c|c|} \hline \cdots & p_{1n} \\ \hline \cdots & p_{2n} \\ \hline \end{array} \in \Theta$$

Tape

$\sharp$	$\sharp$	$\sharp$	$\sharp$	p_{1m+2}	$\cdots$	p_{1n}
p_{11}	$\cdots$	p_{1m}	p_{1m+1}	p_{2m+2}	$\cdots$	p_{2n}

$$\begin{array}{|c|c|}\hline p_{1m+2} & \cdots \\ \hline p_{2m+2} & \cdots \\ \hline \end{array} \in \Theta$$

Tape

$\sharp$	$\sharp$	$\sharp$	p_{1m+1}	p_{1m+2}	$\cdots$	p_{1n}
p_{11}	$\cdots$	p_{1m}	p_{2m+1}	p_{2m+2}	$\cdots$	p_{2n}

$$\begin{array}{|c|c|}\hline p_{1m+1} & p_{1m+2} \\ \hline p_{2m+1} & p_{2m+2} \\ \hline \end{array} \in \Theta$$

Tape

$\sharp$ p_{11}	$\sharp$ $\cdots$	p_{1m} p_{2m}	p_{1m+1} p_{2m+1}	p_{1m+2} p_{2m+2}	$\cdots$ $\cdots$	p_{1n} p_{2n}

$$\begin{array}{|c|c|} \hline p_{1m} & p_{1m+1} \\ \hline p_{2m} & p_{2m+1} \\ \hline \end{array} \in \Theta$$

Tape

$\sharp$ p_{11}	$\cdots$ $\cdots$	p_{1m} p_{2m}	p_{1m+1} p_{2m+1}	p_{1m+2} p_{2m+2}	$\cdots$ $\cdots$	p_{1n} p_{2n}

$$\begin{array}{|c|c|} \hline \cdots & p_{1m} \\ \hline \cdots & p_{2m} \\ \hline \end{array} \in \Theta$$

Tape

p_{11} p_{21}	$\cdots$ $\cdots$	p_{1m} p_{2m}	p_{1m+1} p_{2m+1}	p_{1m+2} p_{2m+2}	$\cdots$ $\cdots$	p_{1n} p_{2n}

$$\begin{array}{|c|c|} \hline p_{11} & \cdots \\ \hline p_{21} & \cdots \\ \hline \end{array} \in \Theta$$

Tape

p_{11} p_{21}	$\cdots$ $\cdots$	p_{1m} p_{2m}	p_{1m+1} p_{2m+1}	p_{1m+2} p_{2m+2}	$\cdots$ $\cdots$	p_{1n} p_{2n}

$$\begin{array}{|c|c|} \hline \sharp & p_{11} \\ \hline \sharp & p_{21} \\ \hline \end{array} \in \Theta$$

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For any unary two-dimensional language L

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- We introduce some two-dimensional languages
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- We introduce some two-dimensional languages
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  - the accepting-computation language of the Turing machine accepting $\phi(L)$
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End of the proof

Accepting computations of a Turing machine

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- Given two configuration C and D we write $C \triangleright D$ whenever we can go from C to D without head reversals
- An accepting computation is a sequence

$$W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$$

  where
  - W_1 is an initial configuration
  - W_n is an accepting configuration
  - at W_i there is a head reversal at W_i

Picture associated with an accepting-computation

Given any accepting computation $W_1 \rhd W_2 \rhd \cdots \rhd W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row

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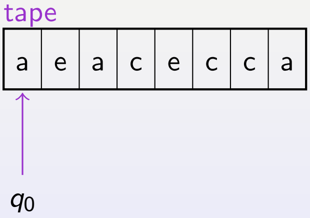

picture

a_0	e	a	c	e	c	c	a

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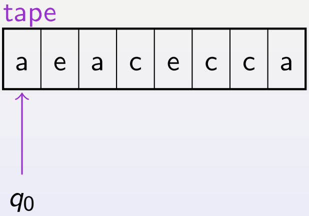

$\delta(q_0, a) \ni (q_1, c, +)$

picture

a_0	e	a	c	e	c	c	a

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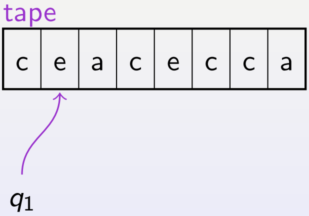
Given any accepting computation $W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row



$$\delta(q_1, e) \ni (q_4, a, +)$$

picture

a_0	e	a	c	e	c	c	a

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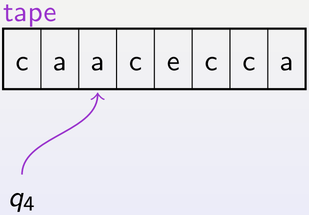
Given any accepting computation $W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row



$$\delta(q_4, a) \ni (q_2, c, +)$$

picture

a_0	e	a	c	e	c	c	a

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Given any accepting computation $W_1 \rhd W_2 \rhd \cdots \rhd W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row

tape

c	a	c	c	e	c	c	a

q_2

$$\delta(q_2, c) \ni (q_1, e, +)$$

picture

a_0	e	a	c	e	c	c	a

Picture associated with an accepting-computation

Given any accepting computation $W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row

tape

c	a	c	e	e	c	c	a

q_1

$$\delta(q_1, e) \ni (q_4, a, +)$$

picture

a_0	e	a	c	e	c	c	a

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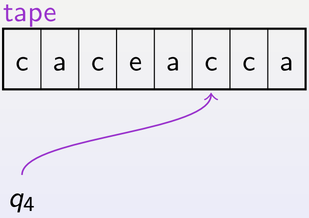

$$\delta(q_4, c) \ni (q_5, e, -)$$

Head reversal!

picture

a_0	e	a	c	e	c	c	a

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tape

c	a	c	e	a	c	c	a

q_4

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Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a

Picture associated with an accepting-computation

Given any accepting computation $W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row

tape

c	a	c	e	a	e	c	a

q_5

$$\delta(q_5, a) \ni (q_1, c, -)$$

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a

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tape

c	a	c	e	c	e	c	a

q_1

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Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a

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tape

c	a	c	a	c	e	c	a

q_4

$\delta(q_4, c) \ni (q_0, e, +)$

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a

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tape

c	a	c	a	e	e	c	a

q_0

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picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a

Picture associated with an accepting-computation

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tape

c	a	c	a	e	a	c	a

q_2

$$\delta(q_2, c) \ni (q_3, e, +)$$

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a

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tape

c	a	c	a	e	a	e	a

q_3

$\delta(q_3, a) \ni (q_2, e, -)$

Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a

Picture associated with an accepting-computation

Given any accepting computation $W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row

tape

c	a	c	a	e	a	e	a

q_3

$\delta(q_3, a) \ni (q_2, e, -)$

Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3

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tape

c	a	c	a	e	a	e	e

q_2

$$\delta(q_2, e) \ni (q_4, a, -)$$

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3

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tape

c	a	c	a	e	a	a	e

q_4

$\delta(q_4, a) \ni (q_2, c, +)$

Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3

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tape

c	a	c	a	e	a	a	e

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Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e

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tape

c	a	c	a	e	c	a	e

q_2

$\delta(q_2, a) \ni (q_0, e, -)$

Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e

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tape

c	a	c	a	e	c	a	e

q_2

$\delta(q_2, a) \ni (q_0, e, -)$

Head reversal!

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e
c	a	c	a	e	c	a_2	e

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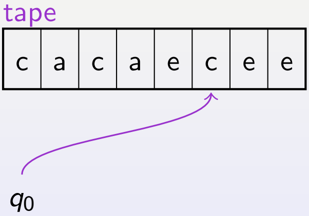

$$\delta(q_0, c) \ni (q_y, e, -)$$

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e
c	a	c	a	e	c	a_2	e

Picture associated with an accepting-computation

Given any accepting computation $W_1 \triangleright W_2 \triangleright \cdots \triangleright W_n$ on input w, let $m = \max |W_i|$ and consider the picture of size $n \times m$ containing W_i on the i-th row

tape

c	a	c	a	e	e	e	e

q_y

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e
c	a	c	a	e	c	a_2	e

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tape

c	a	c	a	e	e	e	e

q_y

picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e
c	a	c	a	e	c	a_2	e
c	a	c	a	e_y	e	e	e

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picture

a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e
c	a	c	a	e	c	a_2	e
c	a	c	a	e_y	e	e	e

Accepting-computation language

The accepting-computation language of M is defined as the set $A(M)$ of all pictures corresponding to any accepting computation of M.

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The accepting-computation language of a Turing machine is in $\textsc{Rec}$

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a_0	e	a	c	e	c	c	a
c	a	c	e	a	c_4	c	a
c	a	c	e_1	c	e	c	a
c	a	c	a	e	a	e	a_3
c	a	c	a	e	a_4	a	e
c	a	c	a	e	c	a_2	e
c	a	c	a	e_y	e	e	e

Accepting-computation language

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$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	a	c	e	a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	e	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

Proposition

The accepting-computation language of a Turing machine is in REC

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The accepting-computation language of a Turing machine is in $\textsc{Rec}$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	a	c	e	a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in $\textsc{Rec}$

tape

a	e	a	c	e	c	c	a

q_0

$$\delta(q_0, a) \ni (q_1, c, +)$$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	a	c	e	a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in $\textsc{Rec}$

tape

c	e	a	c	e	c	c	a

q_1

$$\delta(q_1, e) \ni (q_4, a, +)$$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	c	e	a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in REC

tape

c	a	a	c	e	c	c	a

q_4

$$\delta(q_4, a) \ni (q_2, c, +)$$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	${}^4c$	e	a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in REC

tape

c	a	c	c	e	c	c	a

q_2

$$\delta(q_2, c) \ni (q_1, e, +)$$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	${}^4c$	${}^2e$	a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in $\textsc{Rec}$

tape

c	a	c	e	e	c	c	a

q_1

$\delta(q_1, e) \ni (q_4, a, +)$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	^1a	^4c	^2e	^1a	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in $\textsc{Rec}$

tape

c	a	c	e	a	c	c	a

q_4

$$\delta(q_4, c) \ni (q_5, e, -)$$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	${}^4c$	${}^2e$	${}^1a$	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	c	e	c	a
c	a	c	a	e	a	a	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	a	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	e	e	e

The accepting-computation language is in $\textsc{Rec}$

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	${}^4c$	${}^2e$	${}^1a$	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	${}^5c$	e	c	a
c	a	c	a	${}^4e$	${}^0a$	${}^2e$	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	${}^2a$	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	${}^0e$	e	e

The accepting-computation language is in REC

The marked picture
can be described locally!

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	${}^4c$	${}^2e$	${}^1a$	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	${}^5c$	e	c	a
c	a	c	a	${}^4e$	${}^0a$	${}^2e$	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	${}^2a$	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	${}^0e$	e	e

The accepting-computation language is in $\textsc{Rec}$

The marked picture
can be described locally!

Hence the accepting-computation
language is in $\textsc{Rec}$.

$\overrightarrow{a_0}$	e	a	c	e	c	c	a
c	${}^1a$	${}^4c$	${}^2e$	${}^1a$	$\overleftarrow{c_4}$	c	a
c	a	c	$\overrightarrow{e_1}$	${}^5c$	e	c	a
c	a	c	a	${}^4e$	${}^0a$	${}^2e$	$\overleftarrow{a_3}$
c	a	c	a	e	$\overrightarrow{a_4}$	${}^2a$	e
c	a	c	a	e	c	$\overleftarrow{a_2}$	e
c	a	c	a	$\overrightarrow{e_y}$	${}^0e$	e	e

Overlap of languages

$$p = \begin{array}{|c|c|c|} \hline p_{11} & \cdots & p_{1m} \\ \hline p_{21} & \cdots & p_{2m} \\ \hline \vdots & \cdots & \vdots \\ \hline p_{n1} & \cdots & p_{nm} \\ \hline \end{array}$$

$$q = \begin{array}{|c|c|c|} \hline q_{11} & \cdots & q_{1m} \\ \hline q_{21} & \cdots & q_{2m} \\ \hline \vdots & \cdots & \vdots \\ \hline q_{n1} & \cdots & q_{nm} \\ \hline \end{array}$$

Overlap of languages

Product $p \times q$

$$p = \begin{array}{|c|c|c|}\hline p_{11} & \cdots & p_{1m} \\ \hline p_{21} & \cdots & p_{2m} \\ \hline \vdots & \cdots & \vdots \\ \hline p_{n1} & \cdots & p_{nm} \\ \hline \end{array}$$

$$q = \begin{array}{|c|c|c|}\hline q_{11} & \cdots & q_{1m} \\ \hline q_{21} & \cdots & q_{2m} \\ \hline \vdots & \cdots & \vdots \\ \hline q_{n1} & \cdots & q_{nm} \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline (p_{11}, q_{11}) & \cdots & (p_{1m}, q_{1m}) \\ \hline (p_{21}, q_{21}) & \cdots & (p_{2m}, q_{2m}) \\ \hline \vdots & \cdots & \vdots \\ \hline (p_{n1}, q_{n1}) & \cdots & (p_{nm}, q_{nm}) \\ \hline \end{array}$$

Overlap of languages

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$$p = \begin{array}{|c|c|c|} \hline p_{11} & \cdots & p_{1m} \\ \hline p_{21} & \cdots & p_{2m} \\ \hline \vdots & \cdots & \vdots \\ \hline p_{n1} & \cdots & p_{nm} \\ \hline \end{array}$$

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$$\begin{array}{|c|c|c|} \hline (p_{11}, q_{11}) & \cdots & (p_{1m}, q_{1m}) \\ \hline (p_{21}, q_{21}) & \cdots & (p_{2m}, q_{2m}) \\ \hline \vdots & \cdots & \vdots \\ \hline (p_{n1}, q_{n1}) & \cdots & (p_{nm}, q_{nm}) \\ \hline \end{array}$$

$L_1 \diamond L_2$ is defined as the set of pictures $p_1 \times p_2$, $p_i \in L_i$, such that

- p_1 and p_2 have the same size
- the first row of p_1 equals the first row of p_2

$\phi(L) \in \text{N-LinSpaceRev}_{QU} \Longrightarrow L \in \text{Rec}$

- Assume $\phi(L) \in \text{N-LinSpaceRev}_{QU}$

$\phi(L) \in \textsc{N-LinSpaceRev}_{QU} \Longrightarrow L \in \textsc{Rec}$

- Assume $\phi(L) \in \textsc{N-LinSpaceRev}_{QU}$
- Let A be the accepting-computation language associated with the Turing machine that accept $\phi(L)$

$\phi(L) \in \text{N-LinSpaceRev}_{QU} \Longrightarrow L \in \text{Rec}$

- Assume $\phi(L) \in \text{N-LinSpaceRev}_{QU}$
- Let A be the accepting-computation language associated with the Turing machine that accept $\phi(L)$
- Add paddings to tune the sizes

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$$p \in A$$

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$\phi(L) \in \text{N-LINSPACEREV}_{QU} \Longrightarrow L \in \text{REC}$

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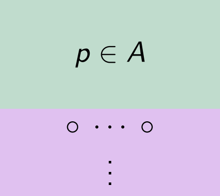
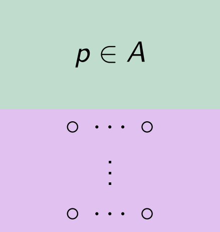


$\phi(L) \in \text{N-LinSpaceRev}_{QU} \Longrightarrow L \in \text{Rec}$

- Assume $\phi(L) \in \text{N-LinSpaceRev}_{QU}$
- Let A be the accepting-computation language associated with the Turing machine that accept $\phi(L)$
- Add paddings to tune the sizes, obtaining $A' = A \ominus \circ^{**}$



$$A' = A \ominus \circ^{**}$$

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- Consider the mask languages

$\phi(L) \in \textsc{N-LinSpaceRev}_{QU} \Longrightarrow L \in \textsc{Rec}$

- Assume $\phi(L) \in \textsc{N-LinSpaceRev}_{QU}$
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- Consider the mask languages M_s

$$\begin{matrix} \circ & \cdots & \circ \\ & \vdots & \\ \circ & \cdots & \circ \end{matrix}$$

$M_s = \{\text{Unary squares}\}$

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- Assume $\phi(L) \in \text{N-LinSpaceRev}_{QU}$
- Let A be the accepting-computation language associated with the Turing machine that accept $\phi(L)$
- Add paddings to tune the sizes, obtaining $A' = A \ominus \circ^{**}$
- Consider the mask languages M_s, M_h

$$\begin{array}{c|c|c} \circ \cdots \circ & h & \circ \cdots \cdots \circ \\ \vdots & h & \vdots \\ \circ \cdots \circ & h & \circ \cdots \cdots \circ \end{array}$$

$$M_h = M_s \varobar h^{*\ominus} \varobar \circ^{**}$$

$\phi(L) \in \text{N-LinSpaceRev}_{QU} \Longrightarrow L \in \text{Rec}$

- Assume $\phi(L) \in \text{N-LinSpaceRev}_{QU}$
- Let A be the accepting-computation language associated with the Turing machine that accept $\phi(L)$
- Add paddings to tune the sizes, obtaining $A' = A \ominus \circ^{**}$
- Consider the mask languages M_s, M_h and M_v

$$\begin{array}{c|c|c} \circ \cdots \circ & v & \circ \cdots \cdots \circ \\ \vdots & v & \vdots \\ \circ \cdots \circ & v & \circ \cdots \cdots \circ \end{array}$$

$$M_v = M_s \obar v^{*\ominus} \obar \circ^{**}$$

$\phi(L) \in \text{N-LinSpaceRev}_{QU} \Longrightarrow L \in \text{Rec}$

- Assume $\phi(L) \in \text{N-LinSpaceRev}_{QU}$
- Let A be the accepting-computation language associated with the Turing machine that accept $\phi(L)$
- Add paddings to tune the sizes, obtaining $A' = A \ominus \circ^{**}$
- Consider the mask languages M_s, M_h and M_v

$$\text{Set } L' = (A' \diamond M_s) \cup (A' \diamond M_h) \cup (A' \diamond M_v)^R$$

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- Consider the mask languages M_s, M_h and M_v

$$\text{Set } L' = (A' \diamond M_s) \cup (A' \diamond M_h) \cup (A' \diamond M_v)^R$$

Then $\pi(L') = L$, where π is the morphism that maps all symbols onto $\circ$.

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$$\text{Set } L' = (A' \diamond M_s) \cup (A' \diamond M_h) \cup (A' \diamond M_v)^R$$

Then $\pi(L') = L$, where π is the morphism that maps all symbols onto $\circ$.

$$\Longrightarrow \; L \text{ is in } \text{Rec}$$

Summary

- Two dimensional tiling-recognizable pictures

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- Representation of two-dimensional unary pictures by quasi-unary strings
- Complexity class for quasi-unary languages: space and number of head reversals bounded.
- Characterization of two-dimensional unary languages in terms of complexity of the corresponding quasi-unary languages