Coven’s Cellular Automata

Gianpiero Cattaneo — Università di Milano–Bicocca

Alberto Dennunzio — Università di Milano–Bicocca

Enrico Formenti — Université de Nice–Sophia Antipolis

Luciano Margara — Università di Bologna

DTDS

A *Discrete Time Dynamical System (DTDS)* is a pair $\langle X, F \rangle$

DTDS

A *Discrete Time Dynamical System (DTDS)* is a pair $\langle X, F \rangle$

- State Space X [Metric Space (X, d)]
- Next State Mapping $F : X \mapsto X$ [Continuous Mapping]

DTDS

A *Discrete Time Dynamical System (DTDS)* is a pair $\langle X, F \rangle$

- State Space X [Metric Space (X, d)]
- Next State Mapping $F : X \mapsto X$ [Continuous Mapping]

Positive motion (or orbit) of initial state $x_0 \in X$ is the sequence $\gamma_{x_0} : \mathbb{N} \mapsto X$ expressed by:

$$\gamma_{x_0} = (x_0, F(x_0), F^2(x_0), \ldots, F^t(x_0), \ldots)$$

$$x_0 \xrightarrow{F} F(x_0) \xrightarrow{F} F^2(x_0) \xrightarrow{F} \cdots F^t(x_0) \xrightarrow{F} \cdots$$

$\forall t \in \mathbb{N},\ \gamma_{x_0}(t) = F^t(x_0)$

DTDS $\left\langle \mathcal{A}^{\mathbb{Z}}, F \right\rangle$ for a finite alphabet $\mathcal{A}$

DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ for a finite alphabet $\mathcal{A}$

- State Space $\mathcal{A}^{\mathbb{Z}}$

An element $\underline{x} \in \mathcal{A}^{\mathbb{Z}}$ is called configuration:

$$\underline{x} = \quad (\ldots \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad \ldots)$$

$$\forall h, k \in \mathbb{Z}, h \leq k, \quad \underline{x}_{[h,k]} = x_h x_{h+1} \cdots x_k \in \mathcal{A}^{k-h+1}$$

DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ for a finite alphabet $\mathcal{A}$

- State Space $\mathcal{A}^{\mathbb{Z}}$

An element $\underline{x} \in \mathcal{A}^{\mathbb{Z}}$ is called configuration:

$$\underline{x} = \quad (\ldots \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad \ldots)$$

$$\forall h, k \in \mathbb{Z}, h \leq k, \quad \underline{x}_{[h,k]} = x_h x_{h+1} \cdots x_k \in \mathcal{A}^{k-h+1}$$

- Tychonoff Metric d_T generating the Product Topology

$$d_T(\underline{x}, \underline{y}) = \sum_{i=-\infty}^{+\infty} \frac{1}{4^{|i|}} d_H(x_i, y_i)$$

d_H: Hamming distance on $\mathcal{A}$

$$\forall n \in \mathbb{N}, \quad d_T(\underline{x}, \underline{y}) < \frac{1}{4^n} \quad \text{iff} \quad \underline{x}_{[-n,n]} = \underline{y}_{[-n,n]}$$

DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ for a finite alphabet $\mathcal{A}$

- State Space $\mathcal{A}^{\mathbb{Z}}$

An element $\underline{x} \in \mathcal{A}^{\mathbb{Z}}$ is called configuration:

$$\underline{x} = \quad (\ldots \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad \ldots)$$

$\forall h, k \in \mathbb{Z}, h \leq k, \quad \underline{x}_{[h,k]} = x_h x_{h+1} \cdots x_k \in \mathcal{A}^{k-h+1}$

- Tychonoff Metric d_T generating the Product Topology

$$d_T(\underline{x}, \underline{y}) = \sum_{i=-\infty}^{+\infty} \frac{1}{4^{|i|}} d_H(x_i, y_i)$$

d_H: Hamming distance on $\mathcal{A}$

$$\forall n \in \mathbb{N}, \quad d_T(\underline{x}, \underline{y}) < \frac{1}{4^n} \quad \text{iff} \quad \underline{x}_{[-n,n]} = \underline{y}_{[-n,n]}$$

$$\begin{array}{ccccccc}
\underline{x}^{(n)} & = & (\ldots & x_{i-1}^{(n)} & x_i^{(n)} & x_{i+1}^{(n)} & \ldots) \\
\downarrow & & & \downarrow & \downarrow & \downarrow & \\
\underline{y} & = & (\ldots & y_{i-1} & y_i & y_{i+1} & \ldots)
\end{array}$$

DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$

DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$

- Next State Mapping $F : \mathcal{A}^{\mathbb{Z}} \mapsto \mathcal{A}^{\mathbb{Z}}$

DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$

- Next State Mapping $F : \mathcal{A}^{\mathbb{Z}} \mapsto \mathcal{A}^{\mathbb{Z}}$
- Space-Time Pattern

Initial configuration $\underline{x}^{(0)} = \underline{x}$

$\forall t \in \mathbb{N}$, $\underline{x}^{(t)} = F^t(\underline{x})$

$$
\begin{array}{|r|cccccccc|c|}
\underline{x} = & (\ldots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \ldots) & & t = 0 \\
F(\underline{x}) = & (\ldots & x_{-2}^{(1)} & x_{-1}^{(1)} & x_0^{(1)} & x_1^{(1)} & x_2^{(1)} & \ldots) & & t = 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\
F^t(\underline{x}) = & (\ldots & x_{-2}^{(t)} & x_{-1}^{(t)} & x_0^{(t)} & x_1^{(t)} & x_2^{(t)} & \ldots) & & t
\end{array}
$$

Cellular Automata (CA)

A One Dimensional Cellular Automaton is a triple $\langle \mathcal{A}, r, f \rangle$

Cellular Automata (CA)

A One Dimensional Cellular Automaton is a triple $\langle \mathcal{A}, r, f \rangle$

$\mathcal{A}$ is a finite *alphabet of symbols*

$r \in \mathbb{N}$ is the *radius* of the automaton,

$f : \mathcal{A}^{2r+1} \mapsto \mathcal{A}$ is the *local rule*

Cellular Automata (CA)

A One Dimensional Cellular Automaton is a triple $\langle \mathcal{A}, r, f \rangle$

- $\mathcal{A}$ is a finite *alphabet of symbols*
- $r \in \mathbb{N}$ is the *radius* of the automaton,
- $f : \mathcal{A}^{2r+1} \mapsto \mathcal{A}$ is the *local rule*

A CA generates a DTDS $\left\langle \mathcal{A}^{\mathbb{Z}}, F_f \right\rangle$

Cellular Automata (CA)

A One Dimensional Cellular Automaton is a triple $\langle \mathcal{A}, r, f \rangle$

$\mathcal{A}$ is a finite *alphabet of symbols*

$r \in \mathbb{N}$ is the *radius* of the automaton,

$f : \mathcal{A}^{2r+1} \mapsto \mathcal{A}$ is the *local rule*

A CA generates a DTDS $\left\langle \mathcal{A}^{\mathbb{Z}}, F_f \right\rangle$

- on the state space $\mathcal{A}^{\mathbb{Z}}$

Cellular Automata (CA)

A One Dimensional Cellular Automaton is a triple $\langle \mathcal{A}, r, f \rangle$

$\mathcal{A}$ is a finite *alphabet of symbols*

$r \in \mathbb{N}$ is the *radius* of the automaton,

$f : \mathcal{A}^{2r+1} \mapsto \mathcal{A}$ is the *local rule*

A CA generates a DTDS $\left\langle \mathcal{A}^{\mathbb{Z}}, F_f \right\rangle$

- on the state space $\mathcal{A}^{\mathbb{Z}}$
- by the next state mapping F_f expressed by the local rule f:

$$\begin{array}{ccc} \underline{x} & = & (\ldots \quad x_{i-r-1} \quad \underbrace{x_{i-r} \ldots x_i \ldots x_{i+r}}_{f} \quad x_{i+r+1} \quad \ldots) \\ F_f(\underline{x}) & = & (\ldots \qquad\qquad [F_f(\underline{x})]_i \qquad\qquad \ldots) \end{array}$$

$[F_f(\underline{x})]_i = f(x_{i-r}, \ldots, x_i, \ldots, x_{i+r})$

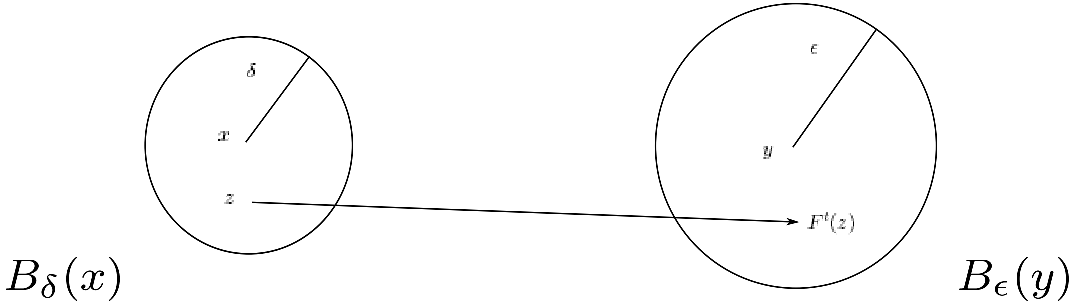
Transitivity, Chain Transitivity

A DTDS $\langle X, F\rangle$ is transitive iff

Transitivity, Chain Transitivity

A DTDS $\langle X, F \rangle$ is transitive iff

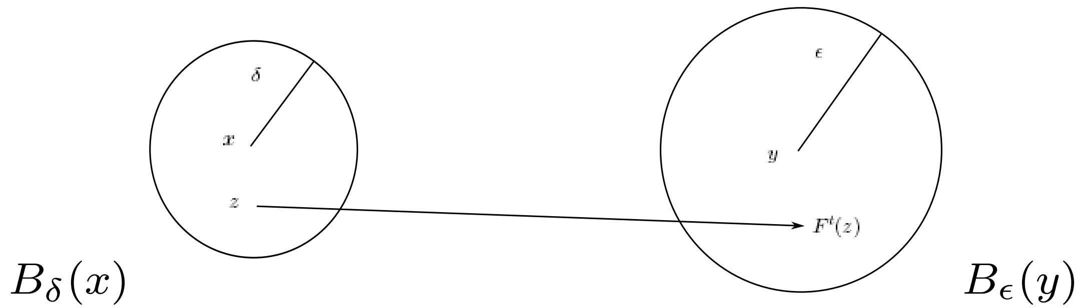
$$\forall x, y \in X, \forall \epsilon, \delta > 0, \exists z \in B_\delta(x), \exists t \in \mathbb{N} : F^t(z) \in B_\epsilon(y)$$



Transitivity, Chain Transitivity

A DTDS $\langle X, F \rangle$ is transitive iff

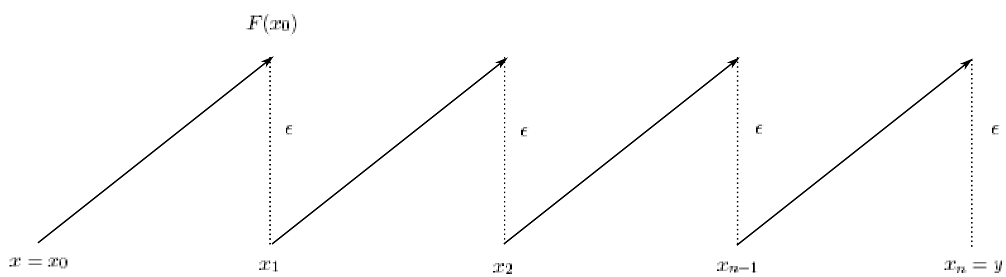
$$\forall x, y \in X, \forall \epsilon, \delta > 0, \exists z \in B_\delta(x), \exists t \in \mathbb{N} : F^t(z) \in B_\epsilon(y)$$



A DTDS $\langle X, F \rangle$ is chain–transitive iff

$\forall x, y \in X, \forall \epsilon > 0$ there exists a ϵ–chain from x to y, i.e.,

$$\exists n > 0, \exists x_0, \ldots, x_n (x_0 = x, x_n = y) : \forall i < n, d(F(x_i), x_{i+1}) < \epsilon$$



Transitivity on $\mathcal{A}^{\mathbb{Z}}$

A DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ is transitive iff
$\forall h, k \in \mathbb{N}, \forall u \in \mathcal{A}^{2h+1}, v \in \mathcal{A}^{2k+1}$,

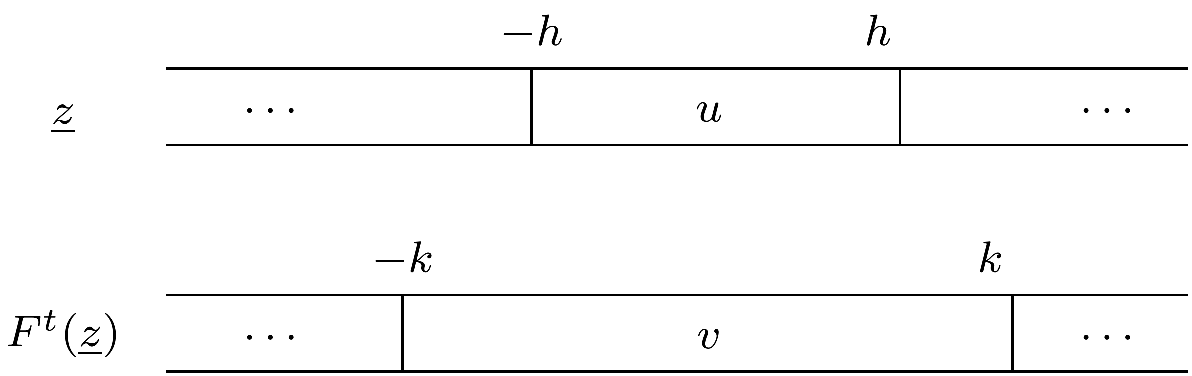
$$\exists \underline{z} \in \mathcal{A}^{\mathbb{Z}}, \underline{z}_{[-h,h]} = u, \exists t \in \mathbb{N} : F^t(\underline{z})_{[-k,k]} = v$$

Transitivity on $\mathcal{A}^{\mathbb{Z}}$

A DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ is transitive iff

$\forall h, k \in \mathbb{N}, \forall u \in \mathcal{A}^{2h+1}, v \in \mathcal{A}^{2k+1},$

$$\exists \underline{z} \in \mathcal{A}^{\mathbb{Z}}, \underline{z}_{[-h,h]} = u, \exists t \in \mathbb{N} : F^t(\underline{z})_{[-k,k]} = v$$



$$\begin{aligned} \underline{z} &= (\ldots \quad \overbrace{z_{-h} \ldots z_0 \ldots z_h}^{u} \quad \ldots) \\ F^t(\underline{z}) &= (\quad \ldots \quad \underbrace{z^t_{-k} \ldots \quad z^t_0 \quad \ldots z^t_k}_{v} \quad \ldots \quad) \end{aligned}$$

Chain Transitivity on $\mathcal{A}^{\mathbb{Z}}$

A DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ is chain-transitive iff
$\forall h \in \mathbb{N}, \forall u, v \in \mathcal{A}^{2h+1}$ there is a chain from u to v, i.e.,

Chain Transitivity on $\mathcal{A}^{\mathbb{Z}}$

A DTDS $\langle \mathcal{A}^{\mathbb{Z}}, F \rangle$ is chain-transitive iff
$\forall h \in \mathbb{N}, \forall u, v \in \mathcal{A}^{2h+1}$ there is a chain from u to v, i.e.,

$$\exists n > 0, \exists u^0, \ldots, u^n (u^0 = u, u^n = v), \exists \underline{x}_0, \ldots, \underline{x}_n (\underline{x}_{0[-h,h]} = u^0) :$$

$$\forall i < n, \quad [F(\underline{x}_i)]_{[-h,h]} = \underline{x}_{i+1[-h,h]} = u^{i+1}$$

		$-h$ … h	
$\underline{x}_0$		$u = u^0$	
$F(\underline{x}_0)$		u^1	
$\underline{x}_1$	$F(\underline{x}_0)$ modificata	u^1	$F(\underline{x}_0)$ modificata
$F(\underline{x}_1)$		u^2	
$\underline{x}_2$	$F(\underline{x}_1)$ modificata	u^2	$F(\underline{x}_1)$ modificata
$\vdots$		$\vdots$	
$F(\underline{x}_{n-1})$		$u^n = v$	
$\underline{x}_n$	$F(\underline{x}_{n-1})$ modificata	$u^n = v$	$F(\underline{x}_{n-1})$ modificata

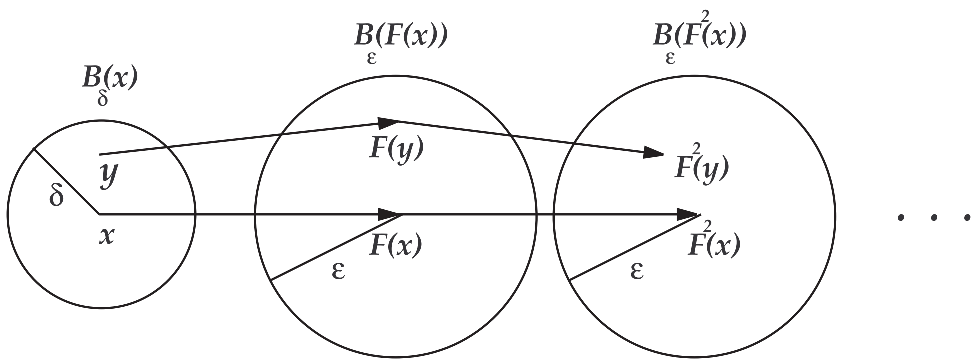
Equicontinuity in DTDS $\langle X, F \rangle$

A state $x \in X$ is equicontinuous (or stable) iff

Equicontinuity in DTDS $\langle X, F \rangle$

A state $x \in X$ is equicontinuous (or stable) iff

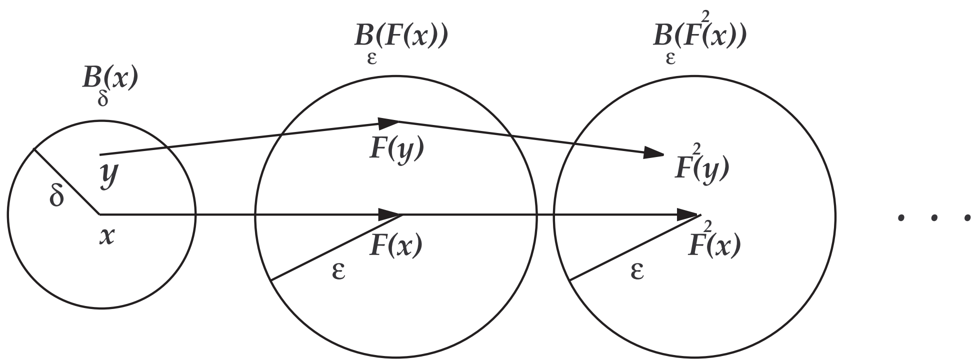
$$\forall \epsilon > 0, \exists \delta > 0 : \forall y \in B_\delta(x) \Rightarrow \forall t \geq 0, F^t(y) \in B_\epsilon(F^t(x))$$



Equicontinuity in DTDS $\langle X, F \rangle$

A state $x \in X$ is equicontinuous (or stable) iff

$$\forall \epsilon > 0, \exists \delta > 0 : \forall y \in B_\delta(x) \Rightarrow \forall t \geq 0, F^t(y) \in B_\epsilon(F^t(x))$$

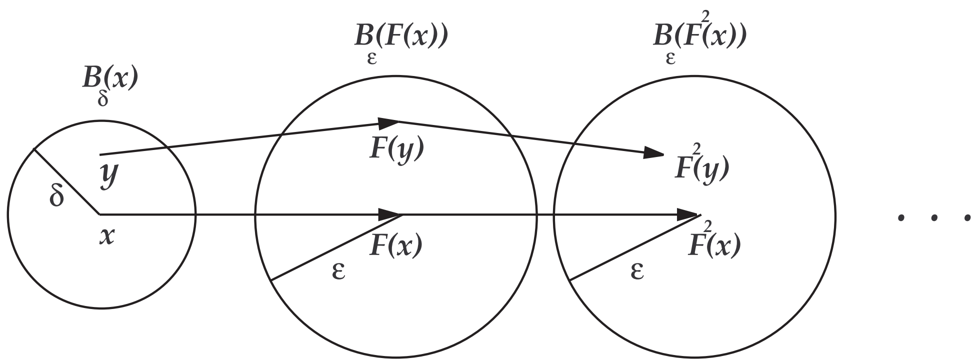


A DTDS is equicontinuous iff the set $\mathcal{E}$ of all equicontinuous states is equal to X: $\mathcal{E} = X$.

Equicontinuity in DTDS $\langle X, F\rangle$

A state $x \in X$ is equicontinuous (or stable) iff

$$\forall \epsilon > 0, \exists \delta > 0 : \forall y \in B_\delta(x) \Rightarrow \forall t \geq 0, F^t(y) \in B_\epsilon(F^t(x))$$



A DTDS is equicontinuous iff the set $\mathcal{E}$ of all equicontinuous states is equal to X: $\mathcal{E} = X$.

A DTDS is almost equicontinuous iff $\mathcal{E}$ is residual (i.e., there is a sequence $\{U_n\}$ of open dense sets, such that $\bigcap_n U_n \subseteq \mathcal{E}$).

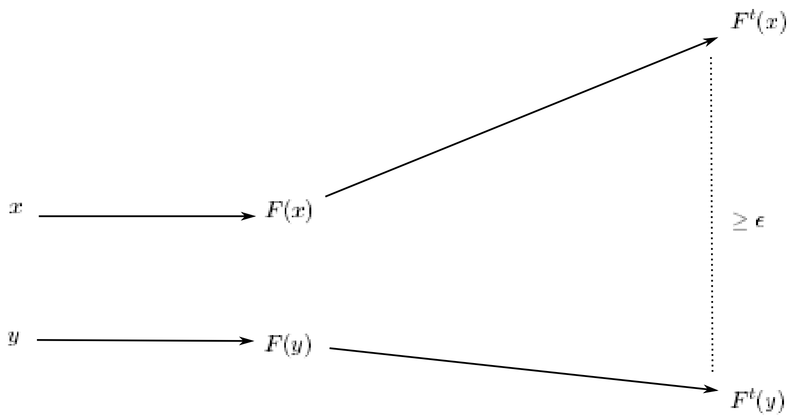
(Positive) Expansivity in DTDS $\langle X, F \rangle$

A DTDS is (positively) expansive iff

(Positive) Expansivity in DTDS $\langle X, F \rangle$

A DTDS is (positively) expansive iff
there exists a constant $\epsilon > 0$ such that

$$\forall x, y \in X, x \neq y, \exists t \in \mathbb{N} : d(F^t(y), F^t(x)) \geq \epsilon$$



Sensitivity in DTDS $\langle X, F \rangle$

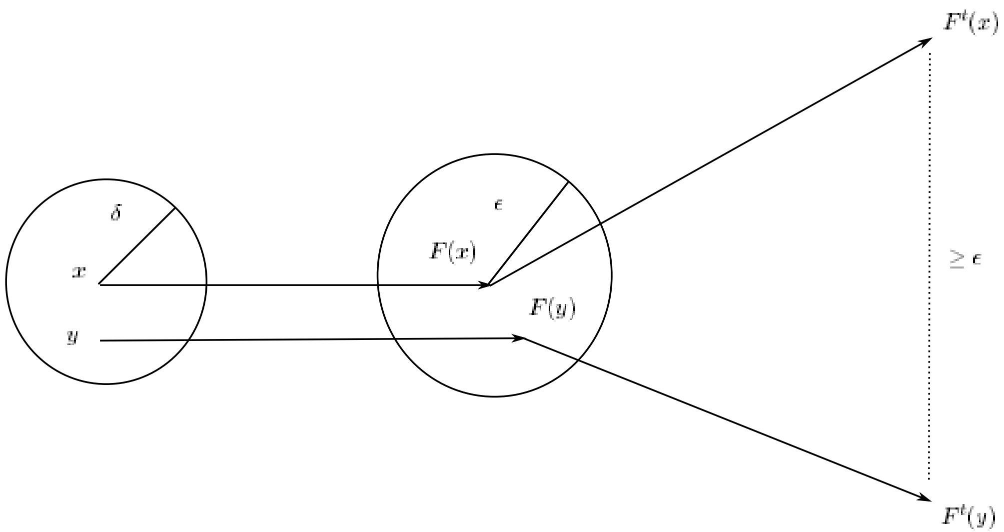
A DTDS is sensitive iff

Sensitivity in DTDS $\langle X, F \rangle$

A DTDS is sensitive iff

there exists a constant $\epsilon > 0$ such that

$$\forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x), \exists t \in \mathbb{N} : d(F^t(y), F^t(x)) \geq \epsilon$$



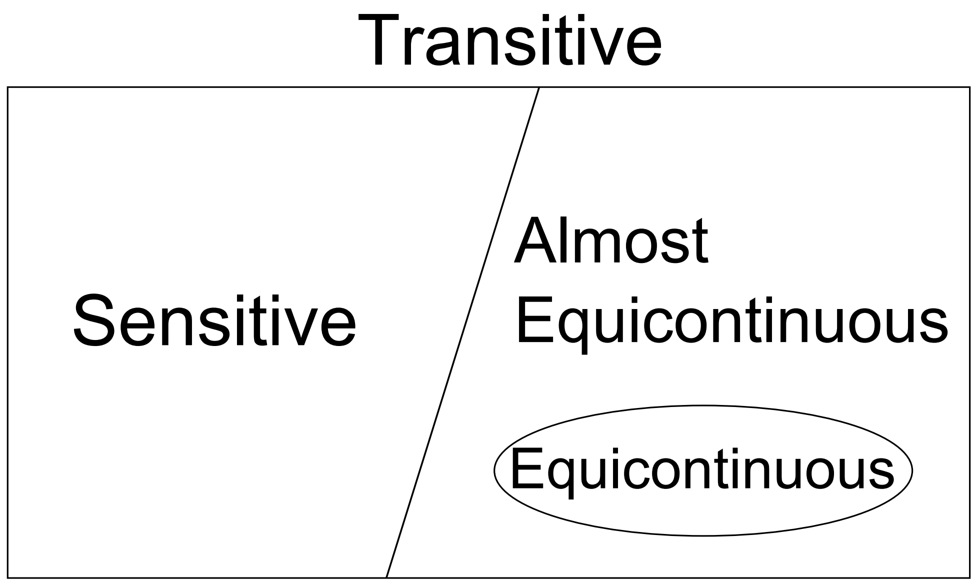
On perfect spaces: Expansivity $\Rightarrow$ Sensitivity

Relationships

For general DTDS:

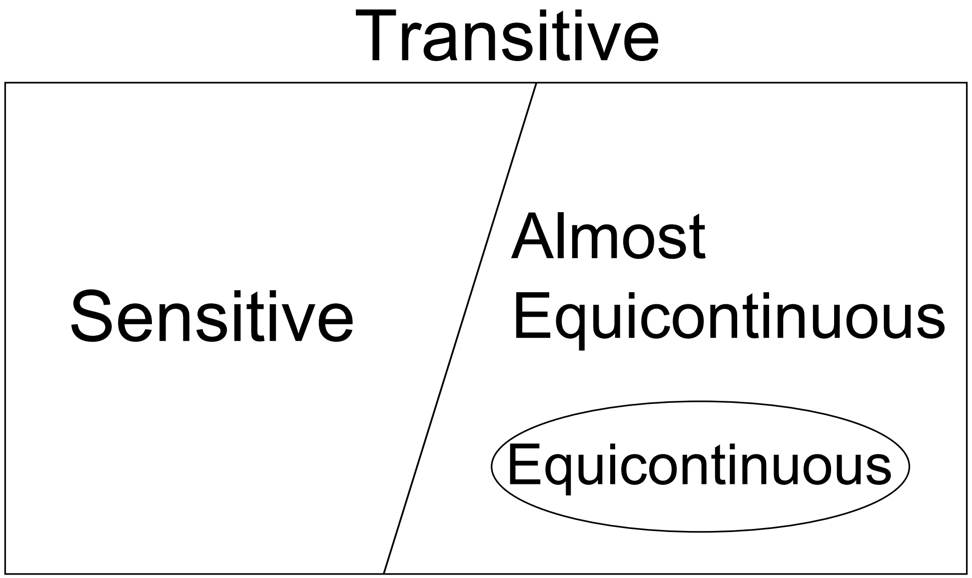
Relationships

For general DTDS:



Relationships

For general DTDS:



For CA: the previous dichotomy is true in general and not only for transitive systems.

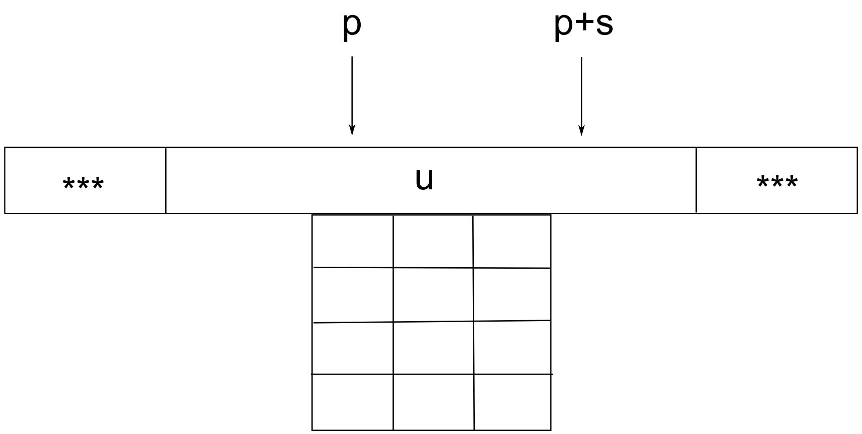
Sensitivity in CA

Let $s > 0$. A word $u \in \mathcal{A}^*$ with $|u| > s$ is s-blocking iff

Sensitivity in CA

Let $s > 0$. A word $u \in \mathcal{A}^*$ with $|u| > s$ is s-blocking iff there exists an offset $p \in \{0, \ldots, |u| - s\}$ such that

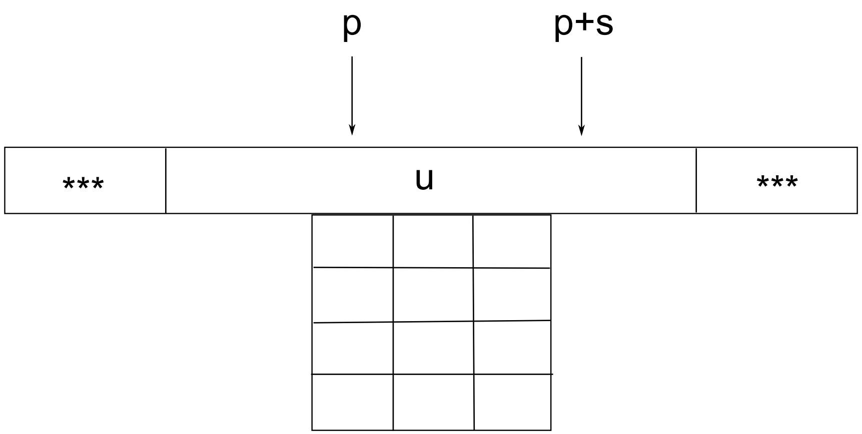
$$\forall \underline{x}, \underline{y}, \quad \underline{x}_{[0,|u|-1]} = \underline{x}_{[0,|u|-1]} = u,$$
$$\forall t \in \mathbb{N}, \quad [F_f^t(\underline{x})]_{[p,p+s-1]} = F_f^t(\underline{y})_{[p,p+s-1]}$$



Sensitivity in CA

Let $s > 0$. A word $u \in \mathcal{A}^*$ with $|u| > s$ is s-blocking iff there exists an offset $p \in \{0, \ldots, |u| - s\}$ such that

$$\forall \underline{x}, \underline{y}, \quad \underline{x}_{[0,|u|-1]} = \underline{x}_{[0,|u|-1]} = u,$$
$$\forall t \in \mathbb{N}, \quad [F_f^t(\underline{x})]_{[p,p+s-1]} = F_f^t(\underline{y})_{[p,p+s-1]}$$



Result: a radius r CA is not sensitive iff it has an r-blocking word iff it is almost equicontinuous.

Coven Automata on $\mathcal{A} = \{0, \ldots, k-1\}$

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a r–block.

The Coven CA local rule is the right-sided mapping $f : \mathcal{A}^{r+1} \mapsto \mathcal{A}$ defined as follows:

Coven Automata on $\mathcal{A} = \{0, \ldots, k-1\}$

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a r–block.

The Coven CA local rule is the right-sided mapping $f : \mathcal{A}^{r+1} \mapsto \mathcal{A}$ defined as follows: $\forall (a_0 a_1 \cdots a_r) \in \mathcal{A}^{r+1}$,

$$f(a_0, a_1, \ldots, a_r) = \begin{cases} \overline{a_0} & \text{if} \quad a_1 \cdots a_r = B \\ a_0 & \text{otherwise} \end{cases}$$

where $\overline{a_0} = (a_0 + 1) \mod k$

Coven Automata on $\mathcal{A} = \{0, \ldots, k-1\}$

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a r–block.

The Coven CA local rule is the right-sided mapping $f : \mathcal{A}^{r+1} \mapsto \mathcal{A}$ defined as follows: $\forall (a_0 a_1 \cdots a_r) \in \mathcal{A}^{r+1}$,

$$f(a_0, a_1, \ldots, a_r) = \begin{cases} \overline{a_0} & \text{if} \quad a_1 \cdots a_r = B \\ a_0 & \text{otherwise} \end{cases}$$

where $\overline{a_0} = (a_0 + 1) \mod k$

$$\begin{array}{lcllllll} \underline{x} & = & (\ldots & x_{i-1} & x_i & \overbrace{x_{i+1} \ldots x_{i+r}}^{=B} & x_{i+r+1} & \ldots) \\ F_f(\underline{x}) & = & (\ldots & & \overline{x_i} & & & \ldots) \end{array}$$

Coven Automata on $\mathcal{A} = \{0, \ldots, k-1\}$

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a r–block.

The Coven CA local rule is the right-sided mapping $f : \mathcal{A}^{r+1} \mapsto \mathcal{A}$ defined as follows: $\forall (a_0 a_1 \cdots a_r) \in \mathcal{A}^{r+1}$,

$$f(a_0, a_1, \ldots, a_r) = \begin{cases} \overline{a_0} & \text{if} \quad a_1 \cdots a_r = B \\ a_0 & \text{otherwise} \end{cases}$$

where $\overline{a_0} = (a_0 + 1) \mod k$

$$\begin{array}{ccccccccc} \underline{x} & = & (\ldots & x_{i-1} & x_i & \overbrace{x_{i+1} \ldots x_{i+r}}^{=B} & x_{i+r+1} & \ldots) \\ F_f(\underline{x}) & = & (\ldots & & \overline{x_i} & & & \ldots) \end{array}$$

$$\begin{array}{ccccccccc} \underline{x} & = & (\ldots & x_{i-1} & x_i & \overbrace{x_{i+1} \ldots x_{i+r}}^{\neq B} & x_{i+r+1} & \ldots) \\ F_f(\underline{x}) & = & (\ldots & & x_i & & & \ldots) \end{array}$$

Period of a word

A word $B = b_1 \cdots b_r \in \mathcal{A}^r$ is periodic of (least) period $p \in \{1, r-1\}$ iff for each $i = 1, \ldots, r-p$, we have $b_i = b_{i+p}$.

If there is no p, the word B is said to be aperiodic.

Period of a word

A word $B = b_1 \cdots b_r \in \mathcal{A}^r$ is periodic of (least) period $p \in \{1, r-1\}$ iff for each $i = 1, \ldots, r-p$, we have $b_i = b_{i+p}$.

If there is no p, the word B is said to be aperiodic.

Example with $\mathcal{A} = \{0, 1\}$:

$B = 111111$ is periodic of period 1.

$B = 101010$ is periodic of period 2.

$B = 100001$ is periodic of period 5.

$B = 111000$ is aperiodic.

Coven Aperiodic Automata

Result:

Coven Aperiodic Automata

Result:

There exists a blocking word and a non-equicontinuous state.

There exists a configuration $\underline{z}$ such that for any $\epsilon > 0$ and any $\underline{x}$, there is a ϵ-chain from $\underline{x}$ to $\underline{z}$ and one from $\underline{z}$ to x.

Coven Aperiodic Automata

Result:

There exists a blocking word and a non-equicontinuous state. There exists a configuration $\underline{z}$ such that for any $\epsilon > 0$ and any $\underline{x}$, there is a ϵ-chain from $\underline{x}$ to $\underline{z}$ and one from $\underline{z}$ to x.

- They are almost equicontinuous (and then not sensitive) without being equicontinuous.
- They are chain transitive but not topologically transitive.

Coven Periodic Automata

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a periodic block of (least) period $p \in \{1, r-1\}$. We can consider three situations:

Coven Periodic Automata

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a periodic block of (least) period $p \in \{1, r-1\}$. We can consider three situations:

1. $p = 1$ (hyper-periodic case)
2. p divides r with $p \neq 1$ (strongly-periodic case)
3. p does not divide r (non factor case)

Coven Periodic Automata

Let $B = b_1 \cdots b_r \in \mathcal{A}^r$ be a periodic block of (least) period $p \in \{1, r-1\}$. We can consider three situations:

1. $p = 1$ (hyper-periodic case)
2. p divides r with $p \neq 1$ (strongly-periodic case)
3. p does not divide r (non factor case)

Example with $\mathcal{A} = \{0, 1\}$ and $r = 6$.

1. Hyper-periodic case: $B = 111111$, $p = 1$.
2. Strongly-periodic case: $B = 101010$, $p = 2$.
3. Non factor case: $B = 100001$, $p = 5$.

The study of the strongly-periodic case can be derived by the hyper-periodic one.

The hyper-periodic case

Result: Hyper-periodic Coven CA are not expansive.

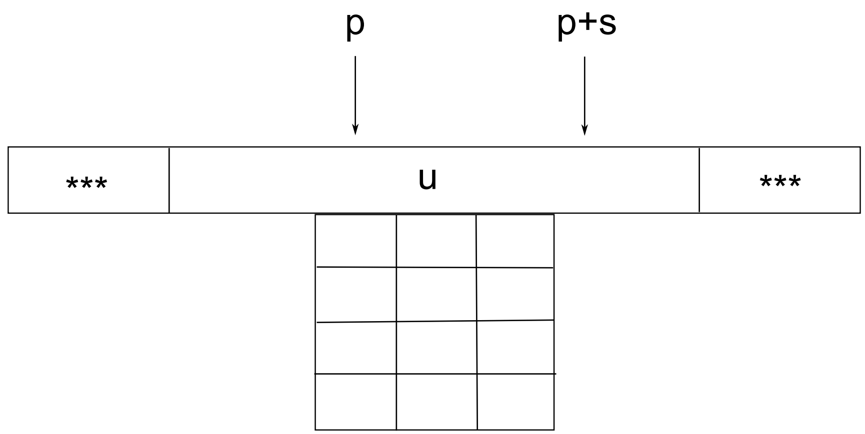
Conjecture: Hyper-periodic Coven CA are sensitive.
Conjecture: Hyper-periodic Coven CA are not chain-transitive.

Hyper-periodic, $B = 11$

$\cdots 00000000$	11	$00 \cdots$	0
$\cdots 00000001$	11	$00 \cdots$	1
$\cdots 00000010$	11	$00 \cdots$	2
$\cdots 00000011$	11	$00 \cdots$	3
$\cdots 00000100$	11	$00 \cdots$	4
$\cdots 00000101$	11	$00 \cdots$	5
$\cdots 00000110$	11	$00 \cdots$	6
$\cdots 00001111$	11	$00 \cdots$	15
$\cdots 00010000$	11	$00 \cdots$	16
$\cdots 00010001$	11	$00 \cdots$	17
$\cdots 00010010$	11	$00 \cdots$	18
$\cdots 00010011$	11	$00 \cdots$	19
$\cdots 00010100$	11	$00 \cdots$	20
$\cdots 00010101$	11	$00 \cdots$	21
$\cdots 00010110$	11	$00 \cdots$	22
$\cdots 00011111$	11	$00 \cdots$	31
$\cdots 00100000$	11	$00 \cdots$	32

Sensitivity, $B = 11$

Combinatorial approach



Topological approach:

$F_f = \sigma \circ F_{130}$