Deterministic recognizability of picture languages by Wang automata*

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Picture languages are a generalization of string languages to two dimensions: a picture is a two-dimensional array of elements from a finite alphabet. Several classes of picture languages have been considered in the literature [5,7,3,10]. In particular, here we refer to class REC introduced in [5] with the aim to generalize to 2D the class of regular string languages. REC is a robust class that has various characterizations; in particular, it is the class of picture languages that can be generated by *tiling systems*, a model introduced in [4], or equivalently by *Wang systems* [**?**].

A central notion in string regular language theory is *determinism*, whereas the concept of determinism for picture languages is far from being well understood. Tiling systems are implicitly nondeterministic: REC is not closed under complement, and the membership problem is NP-complete [8]. Clearly, this latter fact severely hinders the potential applicability of the notation. The identification of a reasonably "rich" deterministic subset of REC would spur its application, since it would allow linear parsing w.r.t. the number of pixels of the input picture.

In past and more recent years, several different deterministic subclasses of REC have been studied, e.g. the classes defined by deterministic 4-way automata [7] or deterministic online tessellation acceptors [6]. This latter model inspired the notion of determinism of [1], that relies on four diagonal-based scanning strategies, each starting from one of the four corners of the picture. Here will call the corresponding deterministic class Diag-DREC[1].

In [9] we introduced the class Snake-DREC, based on a different kind of determinism for tiles, using a boustrophedonic scanning strategy. Snake-DREC properly extends Diag-DREC and is closed under complement, rotation and symmetries. However, like Diag-DREC, it is not closed under intersection and union. When pictures of only one row (or column) are considered, this model reduces to deterministic finite state automata. Quite surprisingly, such notion of determinism coincides with line unambiguity of Row-UREC (or Col-UREC) introduced in [1] to have backtracking at most linear in one dimension of the input picture.

The notion of determinism for tiling systems is quite different than the same notion for 4-way automata [7,5]. For instance, both Snake-DREC and Diag-DREC are incomparable to the class of languages recognized by 4-way deterministic automata. Moreover, any notion of determinism in tiling systems (and online tessellation acceptors) seems to require some pre-established strategy used for scanning the picture.

Indeed, both Diag-DREC and Snake-DREC are based on some fixed kind of strategy. In [2], a first generalization of the concept of scanning strategy is presented; with the same goal, here we propose an alternative framework, where a scanning strategy is defined as a method to sort all cells of a picture,

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[1] The original name is DREC.

in such a way that the next cell to visit is always adjacent to the previous one, and depends only on this information: which neighboring cells have already been visited, and which direction we are moving from. Examples of such scanning strategies are those following the boustrophedonic order, spirals, and many others.

We then introduce a new model of automaton based on tiles. We call such model ***Wang automaton***, since its description is based on the notation of *Wang systems*.

Wang automata combine features of both online tessellation acceptors and 4-ways automata: as in online tessellation acceptors, computation assigns states to each picture position; as in 4-way automata, the input head visits the picture moving from one cell to an adjacent one, according to some scanning strategy. Differently from 4-way automata, Wang automata visit each position exactly once; moreover, one can define various scanning strategies, but next position cannot depend on the input symbol. However, we prove that Wang automata are equivalent to tiling systems, thus they are strictly more powerful than 4-way automata [5].

For Wang automata, it is very natural to introduce determinism. The resulting class properly extends Snake-DREC (and hence Diag-DREC) and is closed under complement, but not intersection and union.

In the rest of this abstract we introduce some notations needed to formally define Wang automata, and we state some basic properties of this model.

Let Σ be a finite alphabet. A two-dimensional array of elements of Σ is a *picture* over Σ. The set of all pictures over Σ is Σ^{++}. A picture language is a subset of Σ^{++}.

Given two positive integer n and m, the *picture domain* of size (n, m) is the set $P = [1..n] \times [1..m]$ and represents a rectangular region with n rows and m columns. Such a region contains *cells*, i.e. unitary squares, and each cell is bordered by 4 *edges*; notice that two adjacent cells share a common edge.

Given a cell c, we use $border(c)$ to denote the set of 4 edges adjacent to c. Dir is the set of directions $\{\mathrm{r}, \mathrm{l}, \mathrm{t}, \mathrm{b}\}$. For every cell c and $d \in Dir$, the edge of c in direction d is denoted by $c|d$, and the cell adjacent to c in direction d is denoted by $c \boxplus d$.

We call *context* any pair in the set $\{(d, D) : d \in D \subset Dir\}$. A context will represent the current direction of the input head of the automaton, together with the information about which neighboring cells have already been visited. The set of contexts is denoted by C. A *next-cell function* is a partial function $\eta : C \to Dir$ such that $\eta(d, D) \notin D$ for every $d \in D \subset Dir$.

Let η be a next-cell function, c_0 a cell, and $d_0 \in Dir$ a direction. Then, for every picture domain P, the triple $\langle \eta, c_0, d_0 \rangle$ induces a path in P starting from c_0 in direction d_0, i.e. the sequence

$$path(P) = c_0, c_1, c_2 \ldots \tag{1}$$

of cells of P satisfying the following inductive definition. Let B_0 be the set of borders (i.e. the set of outer edges) of P and

$$D_i = \{d \in Dir : c_i|d \in B_i\} \qquad B_{i+1} = B_i \cup border(c_i)$$

$$d_{i+1} = \eta(d_i, D_i) \qquad c_{i+1} = c_i \boxplus d_{i+1}$$

In general, the cells appearing in (1) may repeat; moreover the sequence may be infinite. If $path(P)$ is a permutation of P for every picture domain P, then we say that $\langle \eta, c_0, d_0 \rangle$ is a ***picture scanning strategy***. Examples of scanning strategies are those following the boustrophedonic order, spirals, etc.

In [**?**] a variant of Wang tiles equivalent to tiling systems was introduced. A Wang tile is a square tile with colored sides. Color represents compatibility: two tiles may be adjacent only if the color of the touching sides is the same. A *labeled Wang tile* is a Wang tile bearing a *label* in its center; a set of such tiles is called *Wang system*. A rectangular array of labeled Wang tiles generates the picture obtained by taking only the labels of the tiles.

We are now able to formally introduce Wang automata. Let Γ be a set of *colors*. If the edges adjacent to a cell are (partially or fully) colored, a coloring will be used to summarize their colors. Formally, we call *coloring* any partial function $\gamma : Dir \to \Gamma$. The set of directions where γ is defined is denoted by Δ_γ. If $\Delta_\gamma = Dir$, then γ is called a *full* coloring. The set of colorings is denoted by $Colrs$. Given $\gamma_1, \gamma_2 \in Colrs$, we say that γ_2 extends γ_1 if $\gamma_2(d) = \gamma_1(d)$ for every $d \in \Delta_{\gamma_1}$. Any pair $(d, \gamma) \in Dir \times Colrs$ such that $d \in \Delta_\gamma \subset Dir$ is called a *colored context*; the set of colored contexts is denoted by CC.

*A **Wang automaton** consists of a tuple $\langle \Sigma, \Gamma, \eta, c_0, d_0, \delta, F \rangle$, where:*

- *Σ is a finite input alphabet,*
- *Γ is a finite set of colors,*
- *$\langle \eta, c_0, d_0 \rangle$ is a scanning strategy,*
- *F is a set of full colorings over Γ,*
- *$\delta : \Sigma \times CC \to 2^{CC}$ is a function such that*

$$(d', \gamma') \in \delta(\sigma, d, \gamma) \quad \Rightarrow \quad d' = \eta(d, \Delta_\gamma) \text{ and } \gamma' \text{ is a full coloring that extends } \gamma.$$

If $\delta(\sigma, d, \gamma)$ is a singleton for every $(\sigma, d, \gamma) \in \Sigma \times CC$, then the automaton is said to be deterministic.

A Wang automaton can be seen as having a head that visits a picture, by moving from a cell to an adjacent one, and coloring at each step the edges of the cell it is visiting (in a sense, the element of CC are the *states* of the automaton). The movements and coloring operation the automaton performs are determined by a finite control formalized by function δ.

More precisely, the computation of a Wang automaton over an input picture p can be described as follows. At the beginning the head of the automaton points at cell c_0 and the current direction is set to d_0. At step i, let d_i the current direction, c_i be the cell the head is pointing at, σ_i the symbol of p in cell c_i, and γ_i the coloring corresponding to the colors of borders of c_i. Then, if $(d', \gamma') \in \delta(\sigma_i, d_i, \gamma_i)$, the automaton may execute this move: apply γ to the borders of c_i, move to the cell $c_{i+1} = c_i \boxplus d'$, and set the current direction to d'. If no move is possible, the automaton halts. The input picture is accepted if the colors of the border of the final cell correspond to a coloring in F.

Notice that the sequence $c_0, c_1, \ldots, c_m$ of cells visited by the automaton during the computation does not depend on the input symbols and is exactly the path that the scanning strategy $\langle \eta, c_0, d_0 \rangle$ associates with the domain of the input picture p.

Theorem 1. *The class of all picture languages recognized by Wang automata is REC.*

Proposition 1. *The class of all picture languages recognized by deterministic Wang automata includes Snake-DREC and hence Diag-DREC. Such class is closed under complement but not under union nor intersection.*

Proposition 2. *Any fixed strategy defines a sub-class of deterministic Wang automata; the corresponding class of picture languages is a boolean sub-class of REC. In particular, the spiral scanning strategy determines a boolean class that is also closed under rotation and contains picture languages that are not in Snake-DREC.*

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