Catalan structures and Catalan pairs

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Table of contents

Catalan Pairs

Combinatorial interpretations of Catalan pairs
   Perfect noncrossing matchings and Dyck paths
   Plane trees
   Pattern avoiding permutations
   Sequences of integers counted by Catalan numbers
   Staircase shape

Further work
Basic definitions and notations

Catalan numbers are a very popular sequence of integer numbers, arising in many combinatorial problems coming out from different scientific areas, including computer science, computational biology, and mathematical physics.

\[
c_n = \frac{1}{1 + n \binom{2n}{n}}, \forall n \geq 0
\]

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 \ldots

The **Catalan structures** are the combinatorial structures enumerated by Catalan numbers.

The **Catalan pairs** have been introduced with the aim of giving a common language to (almost) all the Catalan structures.

  *Catalan pairs: a relational-theoretic approach to Catalan numbers.*  
Basic definitions and notations

Given any set $X$, we denote $D = D(X) = \{(x, x) \mid x \in X\}$ the diagonal of $X$. Moreover, if $\theta$ is any binary relation on $X$, we denote by $\overline{\theta}$ the symmetrization of $\theta$, i.e. the relation $\overline{\theta} = \theta \cup \theta^{-1}$.

Let $O(X)$ be the set of strict order relations on $X$.

Let $(S, R)$ be an ordered pair of binary relations on $X$. We say that $(S, R)$ is a Catalan pair on $X$ when the following axioms are satisfied:

(i) $S, R \in O(X)$;
(ii) $\overline{R} \cup \overline{S} = X^2 \setminus D$;
(iii) $\overline{R} \cap \overline{S} = \emptyset$;
(iv) $S \circ R \subseteq R$.

$S$ and $R$ are both strict order relations, the two axioms (ii) and (iii) can be explicitly described by saying that, given $x, y \in X$, with $x \neq y$, exactly one of the following holds: $xSy, xRy, ySx, yRx$. The axiom (iv) says that if $xSy$ and $yRz$, then $xRz$. 

Combinatorial properties of $S$ and $R$

The relation $R$ (i.e. the second component) completely defines a Catalan pair, while the relation $S$ does not.

**Theorem**

Let $(S_1, R)$ and $(S_2, R)$ be Catalan pairs, then $S_1 = S_2$.

$$X = \{a, b, c\}$$

$$Y = \{x, y, z\}$$

![Diagrams](image-url)
Combinatorial properties of $S$ and $R$

The set of Catalan Pairs on $X$ is completely determined by the set of relations $R$ on $X$, and the relations $R$, with respect to the size of the base set, are a Catalan structure too.

The posets on 4 elements defined by $R$ are depicted in Figure.

Among the possible 16 nonisomorphic posets on 4 elements, the two missing posets are respectively the poset $2 + 2$ and the poset $Z_4$.

**Proposition**

The relations $R$ are exactly the posets avoiding $2 + 2$ and $Z_4$. 
Combinatorial interpretations of Catalan pairs

General method

Given a Catalan structure $C$, for each object of $C$ we determine a base set $X_C$, and we recursively define a pair of binary relations $(S, R)$ on $X_C$.

Most of the Catalan structures $C$ admit a recursive decomposition as

$$C = \varepsilon + xC \times C$$

meaning that, each element $C \in C$ is the empty object of size zero, or it can be uniquely decomposed as $C = xAB$, where $x$ is an element of unitary size belonging to the base set $X_C$, and $A, B \in C$.

The following Figure shows an example of decomposition for the class of Dyck paths.
Combinatorial interpretations of Catalan pairs

General method

We can recursively define a base set $X_C$ and a Catalan pair $(S, R)$ on $X_C$ for the Catalan structure $C$ in the following way:

- If $C = \varepsilon$ we have $S = \emptyset$ and $R = \emptyset$.
- Otherwise, if $C = xAB$, let $(S_A, R_A)$ and $(S_B, R_B)$ be the Catalan pairs on the objects $A$ and $B$, with base sets $X_A$ and $X_B$, respectively. Then $X_C$ is composed by the elements in $X_A$, $X_B$ plus the new element $x$, and $(S, R)$ is defined as

  $$S = S_A \cup S_B \cup \{(a, x) : a \in X_A\},$$

  $$R = R_A \cup R_B \cup \{(a, b) : a \in X_A, b \in X_B\} \cup \{(x, b) : b \in X_B\}.$$

**Theorem**

The relations $S$ and $R$ form a Catalan pair on the base set $X_C$. 
Combinatorial interpretations of Catalan pairs

Catalan structures in terms of Catalan pairs

We determine the relations $S$ and $R$ for some known Catalan structures involving rather different combinatorial objects:

- Perfect noncrossing matchings and Dyck paths.
- Plane trees.
- Pattern avoiding permutations.
- Sequences of integers counted by Catalan numbers.
- Staircase shape.

- R. P. Stanley, *Catalan addendum*
Perfect noncrossing matchings and Dyck paths

Given a linearly ordered set $A$ of even cardinality, a perfect noncrossing matching of $A$ is a noncrossing partition of $A$ having all the blocks of cardinality 2. A block can be represented by means of an arch joining each couple of points.

Using this representation, we define the following relations on the set $X$ of arches of a given perfect noncrossing matching, for any $x, y \in X$:

- $xSy$ when $x$ is included in $y$;
- $xRy$ when $x$ is on the left of $y$.

Proposition

$(S, R)$ is Catalan pair on the set $X$. 
Perfect noncrossing matchings and Dyck paths

An equivalent way to represent perfect noncrossing matchings is to use Dyck paths: just interpret the leftmost element of an arch as an up step and the rightmost one as a down step.

A tunnel in a Dyck path is a horizontal segment joining the midpoints of an up step and a down step, remaining below the path and not intersecting the path anywhere else.

We define $S$ and $R$ on the set $X$ of the tunnels of a Dyck paths, for any $x, y \in X$:

- $xSy$ when $x$ lies above $y$;
- $xRy$ when $x$ is completely on the left of $y$.

**Proposition**

$(S, R)$ is Catalan pair on the set $X$. 
Perfect noncrossing matchings and Dyck paths

Example

The Catalan pair \((S, R)\) on the set \(X = \{a, b, c, d, e, f, g\}\) is defined as follows:

\[
S = \{(b, a), (f, e), (f, d), (e, d), (g, d)\};
\]

\[
R = \{(a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, e), (b, f), (b, g),
     (c, d), (c, e), (c, f), (c, g), (e, g), (f, g)\}.
\]
Plane trees

Let $T_n$ be the set of plane trees having $n$ edges.

We say that a node $b$ is a descendant of a node $a$ (or in equivalently way $a$ is an ancestor of $b$) when $b$ belongs to the subtree of root $a$.

For any two nodes $b$ and $c$, we define their minimum common ancestor to be the root of the minimum subtree containing both $b$ and $c$ and we say that $b$ lies on the left of $c$ when, called $a$ the minimum common ancestor of $b$ and $c$, $a \notin \{b, c\}$ and $b$ is on the left of $c$.

Given $t \in T_n$, let $X$ denote the set of nodes of $t$ other than the root. The relations $S$ and $R$ on $X$ are defined as follows, for any $x, y \in X$:

- $xSy$ when $x$ is a descendant of $y$;
- $xRy$ when $x$ lies on the left of $y$.

Proposition

$(S, R)$ is Catalan pair on the set $X$.  

The Catalan pair \((S, R)\) on the set \(X = \{a, b, c, d, e, f, g\}\) is defined as follows:

\[
S = \{(b, a), (f, e), (f, d), (e, d), (g, d)\};
\]

\[
R = \{(a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, e), (b, f), (b, g),
(c, d), (c, e), (c, f), (c, g), (e, g), (f, g)\}.
\]
Pattern avoiding permutations

Let $n, m$ be two positive integers with $m \leq n$, and let
$\pi = \pi(1) \cdots \pi(n) \in S_n$ and $\nu = \nu(1) \cdots \nu(m) \in S_m$.

We say that $\pi$ contains the pattern $\nu$ if there exist indices
$i_1 < i_2 < \ldots < i_m$ such that $(\pi(i_1), \pi(i_2), \ldots, \pi(i_m))$ is in the same
relative order as $(\nu(1), \ldots, \nu(m))$. If $\pi$ does not contain $\nu$, we say that $\pi$
is $\nu$-avoiding. For instance, if $\nu = 123$, then $\pi = 524316$ contains $\nu$,
while $\pi = 632541$ is $\nu$-avoiding.

We denote by $S_n(\nu)$ the set of $\nu$-avoiding permutations of $S_n$. For each
pattern $\nu \in S_3$, $|S_n(\nu)| = C_n$

$S_n(312)$ and $S_n(321)$ in terms of Catalan pairs determine a description of
the class $S_n(\nu)$ in terms of Catalan pairs, for any $\nu \in S_3$.

- $S_n(312) \xrightarrow{\text{inverse}} S_n(231) \xrightarrow{\text{symmetry}} S_n(132)$.
- $S_n(312) \xrightarrow{\text{symmetry}} S_n(213)$.
- $S_n(321) \xrightarrow{\text{symmetry}} S_n(123)$. 
312-avoiding permutations

Let $X = \{1, 2, \ldots, n\}$, for every permutation $\pi \in S_n$ we define the following relations $S$ and $R$ on $X$:

- $iSj$ when $i < j$ and $(i, j)$ is an inversion in $\pi$ (i.e. $\pi(i) > \pi(j)$);
- $iRj$ when $i < j$ and $(i, j)$ is a noninversion in $\pi$ (i.e $\pi(i) < \pi(j)$).

Proposition

$(S, R)$ is Catalan pair on the set $X$.

Example

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix}$$

This configuration defines the following Catalan pair $(S, R)$ on the set $X = \{1, 2, \ldots, 6\}$:

$S = \{(1,2), (4,6), (5,6)\}$;

$R = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5)\}$. 
321-avoiding permutations

Given a permutation $\pi = (\pi(1)\ldots\pi(n)) \in S_n$, the element $\pi(i)$ is the point $(i, \pi(i))$ on the Cartesian plane, for any $1 \leq i \leq n$.

Let $X = \{a_1, a_2, \ldots, a_n\}$ be the set of the points on the Cartesian plane representing a permutation $\pi \in S_n(321)$.

If $x, y \in X$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we set:

- $x \prec y$ when $x_1 < y_1$;
- $x \triangleleft y$ when $x_2 < y_2$. 

321-avoiding permutations

Given $x, y \in X$, a *cover* of $\{x, y\}$ is any point $c$ of $X$ having the following properties:

- $x \triangleleft c$ and $y \triangleleft c$;
- $c \prec x$ and $c \prec y$.

For any $x, y \in X$, we say that:

- $xRy$ when there is no cover of $\{x, y\}$, $x \triangleleft y$ and $x \prec y$;
- $xSy$ when $(x, y) \notin \overline{R}$ and $x \prec y$.

The definition of the relation $S$ consists in two distinct cases:

**Proposition**

$(S, R)$ is Catalan pair on the set $X$. 
321-avoiding permutations

Example

\[ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \]

This configuration defines the following Catalan pair \((S, R)\) on the \(X = \{a, b, c, d, e\}\):

\[ S = \{ (a, c), (b, c) \}; \]
\[ R = \{ (a, b), (a, d), (a, e), (b, d), (b, e), (c, d), (c, e), (d, e) \}. \]
Sequences of integers counted by Catalan numbers

The set of sequences $a_1 \ a_2 \ldots \ a_n$ of integers with $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$.

Let $X = \{a_1, a_2, \ldots, a_n\}$ be the set of integers which form the sequence, we define the relations $S$ and $R$ on the set $X$ as follows.

For any $a_i, a_j \in X$, with $1 \leq i, j \leq n$:

- $a_i S a_j$ when $j < i$ and $a_i \leq a_j$;
- $a_i R a_j$ when $i < j$ and $a_i < a_j$.

**Proposition**

$(S, R)$ is Catalan pair on the set $X$. 
Sequences of integers counted by Catalan numbers

Example

\[
\begin{pmatrix}
\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
5 & 2 & 4 & 4 & 5 & 6 \\
\end{pmatrix}
\]

This configuration defines the following Catalan pair \((S, R)\) on the set \(X = \{a_1, a_2, \ldots, a_6\}\):

\[
S = \{(a_2, a_1), (a_3, a_1), (a_4, a_1), (a_5, a_1), (a_4, a_3)\};
\]

\[
R = \{(a_1, a_6), (a_2, a_3), (a_2, a_4), (a_2, a_5), (a_2, a_6), (a_3, a_5), (a_3, a_6), (a_4, a_5),
\]

\[
(a_4, a_6), (a_5, a_6)\}\).
Staircase shape

A staircase shape is depicted in the following Figure.

In particular, \( |b| = |l| = n \) is the size of the staircase shape having \( n \) steps of the form \( \sqcap \).

Two staircase shapes of size \( n \) are said to be different one the other when they are divided into exactly \( n \) rectangles in two different ways.

Proposition

Each staircase shape of size \( n \), with \( n \) rectangles, has \( n \) steps and each step belong to one and only one rectangle.
Staircase shape

Let $X$ be the set of the rectangles which tiles a staircase shape. For any $x \in X$, the sides of the rectangle $x$ are labelled as follows.

At this point, we define the following relations on the set $X$ of rectangles which tile a staircase shape.

For any $x, y \in X$, we set:

- $xSy$ when $l_2(x)$ (or the extension of $l_2(x)$) intersects $b_1(y)$;
- $xRy$ when $x$ is completely on the left of $l_1(y)$ (or the extension of $l_1(y)$).

**Proposition**

$(S, R)$ is Catalan pair on the set $X$. 

\[ 
\begin{array}{c|c}
| & b_2(x) \\
\hline
l_1(x) & x & l_2(x) \\
\hline
b_1(x) & & \\
\end{array} \]
Staircase shape

Example

Let $X = \{a, b, c, d, e, f, g\}$ be the set of the rectangles which tile the staircase shape of size 7 represented in Figure.

This configuration defines the following Catalan pair $(S, R)$ on the set $X = \{a, b, c, d, e, f, g\}$:

$$S = \{(c, d), (c, g), (d, g), (e, g), (f, g)\};$$

$$R = \{(a, b), (a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, g), (b, e), (b, f), (c, e), (c, f), (d, e), (d, f), (e, f)\}.$$
Further work

We adapt our method to the Catalan structures $\mathcal{C}$ which do not admit a recursive decomposition as $\mathcal{C} = \varepsilon + x \mathcal{C} \times \mathcal{C}$ (for instance, parallelogram polyominoes, the 2-colored Motzkin paths, the binary trees).

The class $\mathcal{P}$ of the parallelogram polyominoes admit a recursive decomposition as in the following Figure, where $x$ is a single element of unitary size belonging to the base set $X$ and $A, B, C, D$ are parallelogram polyominoes of lower size.

$$
P = x + x \quad 1 \quad + \quad x \quad 2 \quad + \quad x \quad 3
$$

Also in this case we can recursively define Catalan pairs $(S, R)$ on the class of parallelogram polyominoes.
In particular, if $P$ is a parallelogram polyomino, and it is not the single cell, then it can be uniquely decomposed according to 1, 2 or 3:

1. If the last operation applied on $P$ is operation 1, let $(S_A, R_A)$ be the Catalan pair on the base set for $A$, then $S$ and $R$ are defined as follows:

\[ S = S_A, \quad R = R_A \cup \{(x, a) : a \in A\} \]

2. If the last operation applied on $P$ is operation 2, let $(S_B, R_B)$ be the Catalan pair on the base set for $B$, then $S$ and $R$ are defined as follows:

\[ S = S_B \cup \{(b, x) : b \in B\}, \quad R = R_B. \]

3. If the last operation applied on $P$ is operation 3, let $(S_C, R_C)$, $(S_D, R_D)$ be the Catalan pairs on $C$, $D$, respectively, then $S$ and $R$ are defined as follows:

\[ S = S_C \cup S_D \cup \{(c, x) : c \in C\}, \]

\[ R = R_C \cup R_D \cup \{(c, d) : c \in C, d \in D\} \cup \{(x, d) : d \in D\}. \]
THANKS FOR YOUR ATTENTION