Enumeration of saturated chains in lattices of paths

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Paths

A *class of paths* is the set of all paths starting at the origin of a fixed Cartesian coordinate system, ending on the x-axis, never going below the x-axis, and using only a prescribed set of steps.

E.g.: Dyck paths $\{u, d\}$.

Motzkin paths $\{u, h, d\}$.

Schröder paths $\{u, h^2, d\}$.

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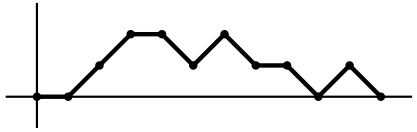
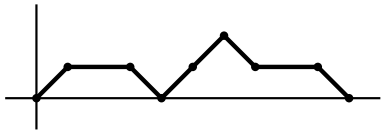
Motzkin paths  $\{u, h, d\}$.

Schröder paths  $\{u, h^2, d\}$.

Paths

Given a class of paths $\mathscr{P}$, the *associated class of Grand paths* is defined in the same way, except for the fact that we are allowed to go below the x-axis.

E.g.: Grand Dyck paths

Grand Motzkin paths

Grand Schröder paths

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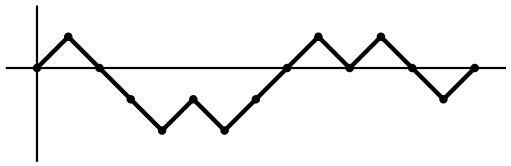
Given a class of paths $\mathscr{P}$, the *associated class of Grand paths* is defined in the same way, except for the fact that we are allowed to go below the x-axis.

E.g.: Grand Dyck paths



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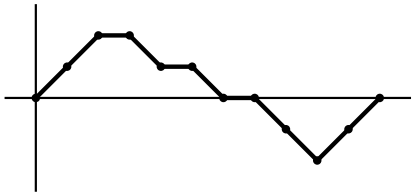

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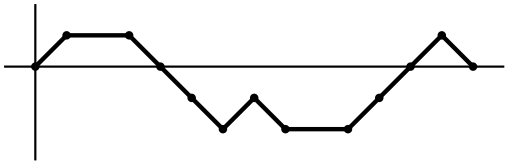

Lattices

Any class of (Grand) paths of the same length can be endowed with a natural partial order structure, by declaring $P \leq Q$ when P lies weakly below Q in the usual two-dimensional drawing of paths.

In several interesting cases (for instance, in all the cases considered above) the resulting poset turns out to be a distributive lattice.

$\mathcal{D}_3$

$\mathcal{M}_4$

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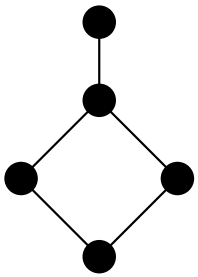

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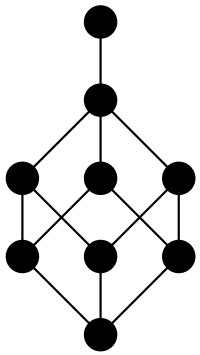

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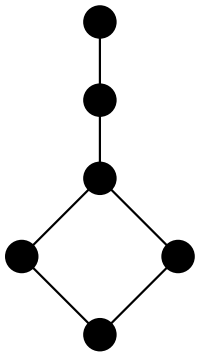

$\mathcal{S}_2$

Our problem

Given a poset, a very natural problem is to count how many saturated chains it has.

A *saturated chain* in a poset is a chain such that, if $x < y$ are consecutive elements in the chain, then y covers x.

We aim at finding a formula for the generating function $SC_h^{\mathscr{P}}(x)$ of saturated chains of length h in the sequence of posets $\mathscr{P}$, for any $h \in \mathbf{N}$.

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Our problem

A related problem concerns the evaluation of the ratio between saturated chains of fixed length and number of elements of the poset.

In the case of saturated chains of length 1, this notion is related to the *density of a graph*, which is an important parameter in the study of topologies for the interconnection of parallel multicomputers, especially in the attempt of finding alternative topologies to the classical one given by the Boolean cube.

Here, we define the *Hasse index of order* k of $\mathcal{P}$ as $i_k(\mathcal{P}) = \frac{sc_k(\mathcal{P})}{|\mathcal{P}|}$, where $sc_k(\mathcal{P})$ denotes the number of saturated chains of length k of the poset $\mathcal{P}$.

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Our problem

We will say that the Hasse index of order k of a sequence $\mathscr{P} = \{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, \ldots\}$ of posets is *Boolean* when $i_k(\mathcal{P}_n) = \frac{(n)_k}{2^k}$, *asymptotically Boolean* when $i_k(\mathcal{P}_n) \sim \frac{(n)_k}{2^k}$ (or, which is the same, $i_k(\mathcal{P}_n) \sim \frac{n^k}{2^k}$) and *asymptotically quasi Boolean* when there exists a small non-negative constant c such that $i_k(P_n) \sim (1/2^k \pm c)n^k$ as $n \to +\infty$. Here, we can assume $c \leq 1/10^k$.

Warm up: saturated chains of length 1

The easiest case occurs when $h = 1$. These are saturated chains of length 1 or, equivalently, edges in the Hasse diagram of the considered posets.

General strategy for counting edges in the Hasse diagram of a sequence of posets $\mathscr{P} = (\mathcal{P}_n)_{n \in \mathbf{N}}$:

- $\ell(\mathcal{P}_n) = \sum_{x \in \mathcal{P}_n} |\Delta x|$, where Δx is the set of all elements covering x in $\mathcal{P}_n$;
- $\ell(\mathcal{P}_n) = \left[\frac{\partial \Delta(\mathcal{P}_n, q)}{\partial q}\right]_{q=1}$, where $\Delta(\mathcal{P}_n, q) = \sum_{x \in \mathcal{P}_n} q^{|\Delta x|}$;
- $\ell_{\mathscr{P}}(x) = \left[\frac{\partial \Delta_{\mathscr{P}}(q, x)}{\partial q}\right]_{q=1}$, where $\ell_{\mathscr{P}}(x) = \sum_n \ell(\mathcal{P}_n) x^n$ and $\Delta_{\mathscr{P}}(q, x) = \sum_n \Delta(\mathcal{P}_n, q) x^n$.

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Dyck and Grand Dyck

Proposition

For any (Grand) Dyck path γ, we have $|\Delta\gamma| = \#(du)_\gamma$, where with $\#(\gamma')_\gamma$ we denote the number of occurrences of the factor γ' in γ.

Dyck and Grand Dyck

Proposition

The generating series for the class of Dyck paths with respect to semi-length (marked by x) and valleys (marked by q) is

$$F(q,x) = \frac{1-(1-q)x-\sqrt{1-2(1+q)x+(1-q)^2x^2}}{2qx}. \tag{1}$$

Proof. $F(q,x) = 1 + xF(q,x) + qxF(q,x)(F(q,x)-1)$

$$F(q,x) = \frac{1-(1-q)x-\sqrt{1-2(1+q)x+(1-q)^2x^2}}{2qx}$$

■

Dyck and Grand Dyck

Theorem

The edge generating series for Dyck lattices is

$$\ell_{\mathcal{D}}(x) = \frac{1 - 3x - (1-x)\sqrt{1-4x}}{2x\sqrt{1-4x}}. \tag{2}$$

Moreover, for every $n \in \mathbf{N}$, $n \geq 2$, the number of edges in $\mathcal{D}_n$ is

$$\ell(\mathcal{D}_n) = \frac{1}{2}\binom{2n}{n}\frac{n-1}{n+1} = \binom{2n-1}{n-2}. \tag{3}$$

In particular, the Hasse index of Dyck lattices is asymptotically Boolean: $i(\mathcal{D}_n) \sim n/2$.

Dyck and Grand Dyck

Proof. Using the above described technique we find (2). Moreover, since $\ell_{\mathcal{D}}(x) = \frac{1+B(x)}{2} - C(x)$, we get (3). ($B(x)$: g.f. of central binomial coefficients; $C(x)$: g.f. of Catalan numbers). Finally $i(\mathcal{D}_n) = \frac{\ell(\mathcal{D}_n)}{|\mathcal{D}_n|} = \frac{n-1}{2} \sim \frac{1}{2}n$. ■

Dyck and Grand Dyck

Proposition

The generating series for the class of Grand Dyck paths with respect to semi-length (marked by x) and valleys (marked by q) is

$$F(q,x) = \frac{1}{\sqrt{1-2(1+q)x+(1-q)^2x^2}}. \tag{4}$$

Dyck and Grand Dyck

Theorem

The edge generating series for Grand Dyck lattices is

$$\ell_{\mathcal{GD}}(x) = \frac{x}{(1-4x)^{3/2}}. \tag{5}$$

Moreover, the number of edges in $\mathcal{GD}_n$ is

$$\ell(\mathcal{GD}_n) = \binom{2n}{n}\frac{n}{2}. \tag{6}$$

In particular, the Hasse index of Grand Dyck lattices is Boolean.

Motzkin and Grand Motzkin

Proposition

For any (Grand) Motzkin path γ, we have
$|\Delta\gamma| = \#(hu)_\gamma + \#(dh)_\gamma + \#(du)_\gamma + \#(hh)_\gamma$.

Motzkin and Grand Motzkin

Proposition

The generating series for the class of Motzkin paths with respect to length (marked by x) and to factors hu, dh, du and hh (marked by q) is

$$F(q,x) = \frac{1 - qx - (1-q)x^2 - \sqrt{(1+x)(1-(1+2q)x-(1-q^2)x^2+(1-q)^2x^3)}}{2qx^2}. \tag{7}$$

Proof. Let $H(q,x)$ be the generating series for the class of Motzkin paths starting with an horizontal step and let $U(q,x)$ be the generating series for the class of Motzkin paths starting with an up step.

$$\begin{cases} F(q,x) = 1 + H(q,x) + U(q,x) \\ H(q,x) = x(1 + qH(q,x) + qU(q,x)) \\ U(q,x) = x^2F(q,x)(1 + qH(q,x) + qU(q,x)) \end{cases}$$

In particular, we obtain expression (7) for $F(q,x)$. ∎

Motzkin and Grand Motzkin

Theorem

The edge generating series for Motzkin lattices is

$$\ell_{\mathcal{M}}(x) = \frac{(1+x)(1-2x-x^2-(1-x)\sqrt{1-2x-3x^2})}{2x^2\sqrt{1-2x-3x^2}}\,. \tag{8}$$

Moreover, the number of edges in $\mathcal{M}_n$ can be expressed in one of the following ways

$$\ell(\mathcal{M}_n) = \binom{n;\,3}{n} - M_n + \binom{n-1;\,3}{n-1} - M_{n-1} \qquad (n \geq 1) \tag{9}$$

$$\ell(\mathcal{M}_n) = \binom{n;\,3}{n-2} + \binom{n-1;\,3}{n-3} \qquad (n \geq 3) \tag{10}$$

$$\ell(\mathcal{M}_n) = \frac{2}{n}\sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{k}\binom{n-k}{k}\frac{k(n-k)}{k+1} \qquad (n \geq 1). \tag{11}$$

In particular, we have the asymptotic expansions

$$\ell(\mathcal{M}_n) \sim \frac{2\cdot 3^n}{\sqrt{3n\pi}} \qquad \text{and} \qquad i(\mathcal{M}_n) = \frac{\ell(\mathcal{M}_n)}{|\mathcal{M}_n|} \sim \frac{4}{9}n \tag{12}$$

and the Hasse index of the Motzkin lattices is asymptotically quasi Boolean.

Motzkin and Grand Motzkin

Proof. Using the above described technique we find (8). Moreover, since $\ell_{\mathcal{M}}(x) = 1 + x(T(x) - M(x))$, we get (9). ($T(x)$: g.f. of central trinomial coefficients, $M(x)$: g.f. of Motzkin numbers). Also, using Cauchy integral formula, we obtain (10). Again, expressing the generating functions $T(x)$ and $M(x)$ in terms of $B(x)$ and $C(x)$, we get (11). Finally, using (9) and the known asymptotic expansions for central trinomial coefficients and Motzkin numbers, we get the asymptotic formulas (12). ■

Motzkin and Grand Motzkin

Proposition

The generating series for the class of Grand Motzkin paths with respect to semi-length (marked by x) and to factors hu, dh, du and hh (marked by q) is

$$F(q,x) = \frac{1+(1-q)x}{\sqrt{(1+x)(1-(1+2q)x-(1-q^2)x^2+(1-q)^2x^3)}}. \quad (13)$$

Motzkin and Grand Motzkin

Theorem

The edge generating series for Grand Motzkin lattices is

$$\ell_{\mathcal{GM}}(x) = \frac{2x^2}{(1-3x)\sqrt{1-2x-3x^2}}. \tag{14}$$

Moreover, we have the identities

$$\ell(\mathcal{GM}_{n+2}) = 2\sum_{k=0}^{n} \binom{k;\ 3}{k} 3^{n-k} \tag{15}$$

$$\ell(\mathcal{GM}_{n+2}) = \frac{2}{4^n}\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}(2k+1)3^k(-1)^{n-k} \tag{16}$$

$$\ell(\mathcal{GM}_{n+2}) = 2\sum_{k=0}^{n} \binom{n+1}{k+1}\binom{2k}{k}(-1)^k 3^{n-k} \tag{17}$$

and the asymptotic equivalences $\ell(\mathcal{GM}_n) \sim 2\cdot 3^{n-2}\sqrt{\frac{3n}{\pi}}$ *and* $i(\mathcal{GM}_n) \sim \frac{4}{9}n$. *In particular, the Hasse index of Grand Motzkin lattices is asymptotically quasi Boolean.*

Schröder and Grand Schröder

Proposition

For any (Grand) Schröder path γ, we have $|\Delta\gamma| = \#(h^2)_\gamma + \#(du)_\gamma$.

Schröder and Grand Schröder

Proposition

The generating series for the class of Schröder paths with respect to semi-length (marked by x) and horizontal steps h and valleys du (marked by q) is

$$F(q,x) = \frac{1 - x - \sqrt{1 - 2(1+2q)x + (1-2q)^2x^2}}{2qx(1+(1-q)x)}. \tag{18}$$

Proof. Let $H(q,x)$ be the generating series for the class of Schröder paths starting with a double horizontal step and let $U(q,x)$ be the generating series for the class of Schröder paths starting with an up step.

$$\begin{cases} F(q,x) = 1 + H(q,x) + U(q,x) \\ H(q,x) = qxF(q,x) \\ U(q,x) = xf(q,x)(1 + h(q,x) + qu(q,x)) \end{cases}$$

In particular, we obtain expression (18) for $F(q,x)$. ■

Schröder and Grand Schröder

Theorem

The edge generating series for Schröder lattices is

$$\ell_{\mathcal{S}}(x) = \frac{(1-x)(1-4x+x^2-(1-x)\sqrt{1-6x+x^2})}{2x\sqrt{1-6x+x^2}}. \tag{19}$$

Moreover, the number of edges in $\mathcal{S}_n$ can be expressed in one of the forms

$$\ell(\mathcal{S}_n) = d_n - r_n - d_{n-1} + r_{n-1} \tag{20}$$

$$\ell(\mathcal{S}_n) = \sum_{k=0}^{n} \left(\!\!\binom{2k}{n-k}\!\!\right) \binom{2k}{k} \frac{k}{k+1}. \tag{21}$$

Finally, we have the asymptotic equivalences

$$\ell(\mathcal{S}_n) \sim \frac{(1+\sqrt{2})^{2n}}{\sqrt{\sqrt{2}n\pi}} \qquad \text{and} \qquad i(\mathcal{S}_n) = \frac{\ell(\mathcal{S}_n)}{|\mathcal{S}_n|} \sim (2-\sqrt{2})n. \tag{22}$$

In particular, the Hasse index of Schröder lattices is asymptotically quasi Boolean.

Schröder and Grand Schröder

Proof. Using the above described technique we find (19). Moreover, since $\ell_{\mathcal{S}}(x) = (1-x)(d(x) - r(x))$, we get (20). ($d(x)$: g.f. of central Delannoy numbers, $M(x)$: g.f. of (large) Schröder numbers). Also, identity (21) is obtained by playing with the same identity $\ell_{\mathcal{S}}(x) = (1-x)(d(x) - r(x))$. Finally, using (20) and the known asymptotic expansions for central Delannoy numbers and Schröder numbers, we get the asymptotic formulas (22). ■

Schröder and Grand Schröder

Proposition

The generating series for the class of Grand Schröder paths with respect to semi-length (marked by x) and horizontal steps h and valleys du (marked by q) is

$$G(q,x) = \frac{1}{\sqrt{1-2(1+2q)x+(1-2q)^2x^2}}. \tag{23}$$

Schröder and Grand Schröder

Theorem

The edge generating series for Grand Schröder lattices is

$$\ell_{\mathcal{GS}}(x) = \frac{2(x-x^2)}{(1-6x+x^2)^{3/2}}. \tag{24}$$

Moreover, we have the identities

$$\ell(\mathcal{GS}_n) = 2\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}(n-k) \tag{25}$$

$$\ell(\mathcal{GS}_n) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\frac{(n-2k)(n+k+2)}{k+1}2^{k+1}3^{n-2k-2} \tag{26}$$

and the asymptotic equivalences $\ell(\mathcal{GS}_n) \sim \sqrt{\frac{n}{2\sqrt{2}\pi}}(1+\sqrt{2})^{2n}$ *and* $i(\mathcal{GS}_n) = \frac{\ell(\mathcal{GS}_n)}{|\mathcal{GS}_n|} \sim (2-\sqrt{2})n$. *In particular, the Hasse index of Grand Schröder lattices is asymptotically quasi Boolean.*

Saturated chains of arbitrary length in Dyck lattices

Let $\gamma < \gamma'$ be two Dyck paths having the same length. A saturated chain starting from γ and ending at γ' is essentially equivalent to a certain set of *skew Young tableaux*.

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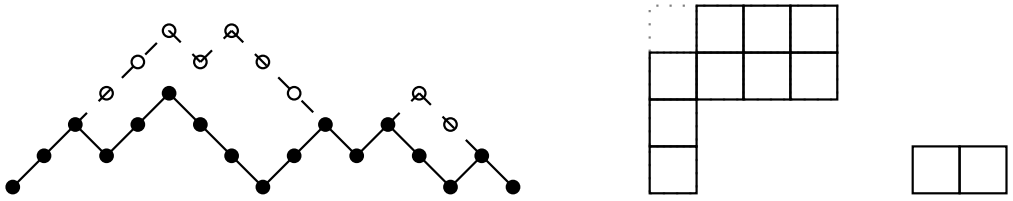

General enumeration formula

How many saturated chains of length h are there in $\mathcal{D}_n$ starting from a given path γ?

- Start with a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of h.
- Next choose a set $\gamma_1, \ldots, \gamma_k$ of *pairwise disjoint* factors of γ such that, for any $i \leq k$, we can build a skew Ferrers shape φ_i on γ_i having area λ_i.
- Finally, linearly order the cells of the Ferrers shapes thus obtained, or, equivalently, endow each of the shapes with a skew Young tableaux structure.

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General enumeration formula

How many saturated chains of length h are there in $\mathcal{D}_n$ starting from a given path γ?

- Start with a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of h.
- Next choose a set $\gamma_1, \ldots, \gamma_k$ of *pairwise disjoint* factors of γ such that, for any $i \leq k$, we can build a skew Ferrers shape φ_i on γ_i having area λ_i.
- Finally, linearly order the cells of the Ferrers shapes thus obtained, or, equivalently, endow each of the shapes with a skew Young tableaux structure.

General enumeration formula

Since the set of integers actually used to fill in the cells of each φ_i can be any possible set of $|\lambda_i|$ integers less than or equal to h, we have proved the following:

Theorem

The number $sc_h(\mathcal{D}_n)$ of saturated chains of length h of the lattice $\mathcal{D}_n$ is given by the following formula:

$$\sum_{\gamma\in\mathcal{D}_n}\sum_{\lambda\vdash h}\sum_{\substack{\gamma_1,\dots,\gamma_k \text{ p.d.o.}\\ (\forall i)(\exists\varphi_i\in SkFS(\lambda_i))b(\varphi_i)=\gamma_i}}\sum_{\substack{(\varphi_1,\dots,\varphi_k)\in SkFS^k\\ (\forall i)(b(\varphi_i)=\gamma_i,A(\varphi_i)=\lambda_i)}}\binom{h}{A(\varphi_1),\dots,A(\varphi_k)}t(\varphi_1)\cdots t(\varphi_k). \tag{27}$$

Saturated chains of length 2

In order to apply formula (27) to the case of saturated chains of length 2 we simply have to set $h = 2$.

Observe that there are only two partitions of 2, namely (1,1) and (2), and there exists one pair of "admissible" skew Ferrers shapes of area 1, i.e. $(\square, \square)$, and two different skew Ferrers shapes of area 2, i.e. $\square\square$ and $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$.

Since each of these shapes can be endowed with only one Young tableau structure, we arrive at the next result (where we have set $SC_2(x) = SC_2^{\mathscr{D}}(x)$).

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The generating series

Proposition

The generating series for the number of saturated chains of length 2 of Dyck lattices is given by

$$SC_2(x) = \sum_{n \geq 0} \left(\sum_{\gamma \in \mathcal{D}_n} (2 \cdot \#(du, du)_\gamma + \#(ddu)_\gamma + \#(duu)_\gamma) \right) x^n, \quad (28)$$

where with $\#(\gamma_1, \ldots, \gamma_k)_\gamma$ we denote the number of pairwise disjoint occurrences of the γ_i's in γ.

All we have to do now is to evaluate the three unknown quantities appearing in (28).

The generating series

Proposition

Denote with $F(q,x)$ and $V(q,x)$ the generating series of all Dyck paths where x keeps track of the semilength and q keeps track of the factor duu and of the factor du (i.e. valleys), respectively. Then

$$SC_2(x) = 2 \cdot \left[\frac{\partial F}{\partial q}\right]_{q=1} + \left[\frac{\partial^2 V}{\partial q^2}\right]_{q=1}. \tag{29}$$

Proof. The expression $\left[\frac{\partial^2 V}{\partial q^2}\right]_{q=1}$ gives the generating series of Dyck paths with respect to the number of (non ordered) pairs of valleys.

Moreover, the expression $\left[\frac{\partial F}{\partial q}\right]_{q=1}$ gives the generating series of Dyck paths with respect to the number of factors duu.

Since the factors ddu and duu are obviously equidistributed in the set of Dyck paths, formula (29) immediately follows.

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The generating series

Theorem

The generating series for the number of saturated chains of length 2 of Dyck lattices is given by

$$SC_2(x) = \frac{1 - 6x + 6x^2 - (1-4x)\sqrt{1-4x}}{-(1-4x)\sqrt{1-4x}}. \tag{30}$$

Consequently

$$sc_2(\mathcal{D}_n) = \binom{2n}{n}\frac{(n-1)(n-2)}{2(2n-1)} \qquad (n \geq 1). \tag{31}$$

Sloane...

Remark. The integer sequence associated with $SC_2(x)$ starts $0, 0, 0, 4, 30, 168, 840, 3960, 18018, 80080, \ldots$. Observe that the terms of the above sequence divided by 2 yield sequence A002740 of the Sloane Encyclopedia. In terms of Dyck paths, this sequence gives the sum of the abscissae of the valleys in all Dyck paths of semilength $n-1$. It would be nice to have a combinatorial explanation of this fact.

Hasse index of order 2

Proposition

The Hasse index of order 2 of the class of Dyck lattices is asymptotically Boolean.

Proof. Since $|\mathcal{D}_n| = \binom{2n}{n}\frac{1}{n+1}$, from formula (31) we get

$$i_2(\mathcal{D}_n) = \frac{sc_2(\mathcal{D}_n)}{|\mathcal{D}_n|} = \frac{(n-1)(n-2)(n+1)}{2(2n-1)} \sim \frac{n^2}{4}.$$

■

Saturated chains of length 3

Setting $h = 3$ in (27), we obtain a formula for the enumeration of saturated chains of length 3 of Dyck lattices.

There are three partitions of the integer 3, namely (1,1,1), (2,1) and (3). Moreover, the unique "admissible" triple of skew Ferrers shapes of area 1 is $(\square, \square, \square)$, whereas there are two pairs of skew Ferrers shapes whose first component have area 1 and whose second component has area 2, namely $(\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix})$ and $(\square, \square\square)$, and there are four skew Ferrers shapes having area 3, i.e. $\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}$, $\square\square\square$, $\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}$ and $\begin{smallmatrix}\square & \square\\ \square & \end{smallmatrix}$.

Unlike the previous case, now we have two shapes (of area 3) each of which can be endowed with two different Young tableaux structures. More precisely, we have to consider the two skew Young tableaux $\begin{smallmatrix} & 3\\ 2 & 1\end{smallmatrix}$, $\begin{smallmatrix} & 2\\ 3 & 1\end{smallmatrix}$ and the two (skew) Young tableaux $\begin{smallmatrix}3 & 1\\ 2 & \end{smallmatrix}$, $\begin{smallmatrix}3 & 2\\ 1 & \end{smallmatrix}$. Thus, a direct application of formula (27) leads to the following statement.

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The generating series

Proposition

The generating series for the number of saturated chains of length 3 of Dyck lattices is given by

$$SC_3(x) = \sum_{n\geq 0} \sum_{\gamma\in\mathcal{D}_n} (6\cdot\#(du,du,du)_\gamma + 3\cdot\#(du,ddu)_\gamma + 3\cdot\#(du,duu)_\gamma + \#(dddu)_\gamma + \#(duuu)_\gamma + 2\cdot\#(dduu)_\gamma + 2\cdot\#(dudu)_\gamma)\, x^n. \tag{32}$$

Our next step will be the evaluation of the unknown quantities appearing in (32).

The generating series

Proposition

Denote with $A(q,x)$, $B(q,x)$ and $C(q,x)$ the generating series of Dyck paths where x keeps track of the semilength and q keeps track of the factors $dduu, dudu$ and $duuu$, respectively. Moreover, let $V(q,x)$ be defined as in the previous section. Finally, let $F(y,q,x)$ be the generating series of Dyck paths obtained from the series $F(q,x)$ defined in the previous section by adding the indeterminate y keeping track of valleys (i.e. of the factor du). Then

$$\begin{aligned} SC_3(x) &= 2\cdot\left[\frac{\partial A}{\partial q}\right]_{q=1} + 2\cdot\left[\frac{\partial B}{\partial q}\right]_{q=1} + 2\cdot\left[\frac{\partial C}{\partial q}\right]_{q=1} \\ &\quad + \left[\frac{\partial^3 V}{\partial q^3}\right]_{q=1} + 6\cdot\left[\frac{\partial^2 F}{\partial y\partial q} - \frac{\partial F}{\partial q}\right]_{y=q=1}. \end{aligned} \tag{33}$$

The generating series

Proof. The meaning of the partial derivatives of the generating series A, B and C is obvious (notice, in particular, that the the factors $dddu$ and $duuu$ are clearly equidistributed, so they are both described by series C).

Moreover, the triple partial derivative of V evaluated in $q = 1$ gives 6 times the distribution of triples of valleys in Dyck paths.

Finally, if we differentiate F with respect to y and q and then evaluate at $y = q = 1$, we obtain the generating series of Dyck paths with respect to semilength and number of pairs (du, duu) (which is equivalent to number of pairs (du, ddu)). However, in this way we are going to consider also those pairs in which the valley du is part of the factor duu. Thus, to obtain what we need, we have to subtract the derivative of F with respect to q, then evaluate at $y = q = 1$, which yields the expression $\left[\frac{\partial^2 F}{\partial y \partial q} - \frac{\partial F}{\partial q}\right]_{y=q=1}$. ∎

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The generating series

Theorem

The generating series for the number of saturated chains of length 3 of $\mathcal{D}_n$ is given by

$$SC_3(x) = \frac{P(x) - Q(x)\sqrt{1-4x}}{x(1-4x)^3}, \tag{34}$$

where
$P(x) = 1 - 13x + 59x^2 - 100x^3 + 16x^4 + 64x^5 = (1-4x)^3(1-x-x^2)$
and $Q(x) = 1 - 11x + 39x^2 - 40x^3 - 22x^4$.

Hasse index of order 3

Proposition

The Hasse index of order 3 of the class of Dyck lattices is asymptotically Boolean.

Proof. Since series (34) can be rewritten as:

$$\frac{1}{x}\left(1 - x - x^2 - \frac{Q(x)}{(1-4x)^{5/2}}\right),$$

when $n > 2$ we have $sc_3(\mathcal{D}_n) = [x^n]SC_3(x) = -[x^{n+1}]Q(x)(1-4x)^{-5/2}$. Using Darboux's theorem, we get

$$sc_3(\mathcal{D}_n) \sim -\frac{Q(\xi)}{\xi^{n+1}}\,\frac{(n+1)^{5/2-1}}{\Gamma(5/2)},$$

where $\xi = \frac{1}{4}$. Since $Q(\xi) = \frac{3}{128}$ and $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$, we obtain $sc_3(\mathcal{D}_n) \sim \frac{2^{2n-3}n^{3/2}}{\sqrt{\pi}}$. Recalling that $|\mathcal{D}_n| \sim \frac{4^n}{n\sqrt{n\pi}}$, we finally have $i_3(\mathcal{D}_n) = \frac{sc_3(\mathcal{D}_n)}{|\mathcal{D}_n|} \sim \frac{n^3}{8}$.

Further work

When h becomes bigger, computations become much more complicated. Is it possible to conceive a different approach more suitable for effective computation?

We have proved that the Hasse indexes of order 1,2 and 3 of Dyck lattices are asymptotically Boolean. The obvious conjecture is that the Hasse index of any order h is asymptotically Boolean too.

Is it possible to extend our approach to enumerate chains in Dyck lattices?

The problem of enumerating (saturated) chains can also be posed for other classes of posets. In this context, it would be interesting to find analogous results in the case of Motzkin and Schröder lattices.

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