REGULARITY OF LANGUAGES DEFINED BY FORMAL SERIES WITH ISOLATED CUT POINT

UNIVERSITÀ DEGLI STUDI DI MILANO

Alberto Bertoni
Maria Paola Bianchi
Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano

SAPIENZA UNIVERSITÀ DI ROMA

Flavio D'Alessandro
Dipartimento di Matematica
Università di Roma "La Sapienza"

In Third Workshop on Non-Classical Models of Automata and Applications (NCMA), Milan, Italy, 2011.

SUMMARY

- Formal power series and languages
- Conditions of regularity for languages described by formal series (rational + Hadamard quotient)
- Hadamard quotient of rational series and two-way weighted automata

FORMAL POWER SERIES

- Formal power series $\varphi \in \mathbb{R}\langle\langle \Sigma^* \rangle\rangle$

$$\varphi = \sum_{\omega \in \Sigma^*} \varphi(\omega)\omega$$

FORMAL POWER SERIES

- **Formal power series** $\varphi \in \mathbb{R}\langle\langle\Sigma^*\rangle\rangle$

$$\varphi = \sum_{\omega \in \Sigma^*} \varphi(\omega)\omega$$

- Polynomial $\varphi \in \mathbb{R}\langle\Sigma^*\rangle$ iff

$$supp(\varphi) = \{\omega \in \Sigma^* | \varphi(\omega) \neq 0\} \text{ is finite}$$

FORMAL POWER SERIES

- **Formal power series** $\varphi \in \mathbb{R}\langle\langle \Sigma^* \rangle\rangle$

$$\varphi = \sum_{\omega \in \Sigma^*} \varphi(\omega)\omega$$

- Polynomial $\varphi \in \mathbb{R}\langle \Sigma^* \rangle$ iff

$$supp(\varphi) = \{\omega \in \Sigma^* | \varphi(\omega) \neq 0\} \text{ is finite}$$

  - sum: $(\varphi + \psi)(\omega) = \varphi(\omega) + \psi(\omega)$
  - Cauchy product: $(\varphi \cdot \psi)(\omega) = \sum_{\substack{x,y \in \Sigma^* \\ xy = \omega}} \varphi(x)\psi(y)$

FORMAL POWER SERIES

- **Formal power series** $\varphi \in \mathbb{R}\langle\langle\Sigma^*\rangle\rangle$

$$\varphi = \sum_{\omega \in \Sigma^*} \varphi(\omega)\omega$$

- Polynomial $\varphi \in \mathbb{R}\langle\Sigma^*\rangle$ iff

$$supp(\varphi) = \{\omega \in \Sigma^* | \varphi(\omega) \neq 0\} \text{ is finite}$$

  - sum: $(\varphi + \psi)(\omega) = \varphi(\omega) + \psi(\omega)$
  - Cauchy product: $(\varphi \cdot \psi)(\omega) = \sum_{\substack{x,y \in \Sigma^* \\ xy = \omega}} \varphi(x)\psi(y)$
  - if $\varphi(\epsilon) = 0$, $\varphi^* = \sum_{i \geq 0} \varphi^i$

FORMAL POWER SERIES

- **Formal power series** $\varphi \in \mathbb{R}\langle\langle\Sigma^*\rangle\rangle$

$$\varphi = \sum_{\omega \in \Sigma^*} \varphi(\omega)\omega$$

- Polynomial $\varphi \in \mathbb{R}\langle\Sigma^*\rangle$ iff

$$supp(\varphi) = \{\omega \in \Sigma^* | \varphi(\omega) \neq 0\} \text{ is finite}$$

  - sum: $(\varphi + \psi)(\omega) = \varphi(\omega) + \psi(\omega)$
  - Cauchy product: $(\varphi \cdot \psi)(\omega) = \sum_{\substack{x,y \in \Sigma^* \\ xy = \omega}} \varphi(x)\psi(y)$
  - if $\varphi(\epsilon) = 0$, $\varphi^* = \sum_{i \geq 0} \varphi^i$

- **Rational series:** $\mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle = (\mathbb{R}\langle\Sigma^*\rangle)^{+,\cdot,*}$

FORMAL POWER SERIES

- Recognizable series $\varphi \in \mathbb{R}^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle$

FORMAL POWER SERIES

- Recognizable series $\varphi \in \mathbb{R}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ iff admits a linear representation:

$$(\pi, \mu, \eta)$$

FORMAL POWER SERIES

- Recognizable series $\varphi \in \mathbb{R}^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$

iff admits a linear representation:

$$\in \mathbb{R}^{1\times m}$$

$$(\pi, \mu, \eta)$$

FORMAL POWER SERIES

- Recognizable series $\varphi \in \mathbb{R}^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$

iff admits a linear representation:

$$\in \mathbb{R}^{1\times m}$$

$$(\pi, \mu, \eta)$$

$$\mu(\sigma) \in \mathbb{R}^{m\times m}$$

FORMAL POWER SERIES

- **Recognizable series** $\varphi \in \mathbb{R}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$

iff admits a **linear representation**:

$$\in \mathbb{R}^{1\times m}$$

$$(\pi, \mu, \eta)$$

$$\mu(\sigma) \in \mathbb{R}^{m\times m}$$

such that for all $\omega = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^*$

$$\varphi(\omega) = \pi\mu(\omega)\eta^T = \pi\left(\prod_{i=1}^n \mu(\sigma_i)\right)\eta^T$$

FORMAL POWER SERIES

- **Recognizable series** $\varphi \in \mathbb{R}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$

  iff admits a **linear representation**:

$$\in \mathbb{R}^{1\times m}$$

$$(\pi, \mu, \eta)$$

$$\mu(\sigma) \in \mathbb{R}^{m\times m}$$

such that for all $\omega = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^*$

$$\varphi(\omega) = \pi\mu(\omega)\eta^T = \pi\left(\prod_{i=1}^n \mu(\sigma_i)\right)\eta^T$$

- **(Schützenberger, 61)**: $\mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle = \mathbb{R}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$

PROOF

- Matrix representation of a 1pfa:

$$A = (\pi, \mu, \eta)$$

initial distribution

stochastic transition matrix

final states

- A is the linear representation of

$$p_A : \Sigma^* \to [0, 1]$$

where $p_A(\omega)$ is the probability of accepting the word ω

LANGUAGES AND FORMAL SERIES

- Language defined by φ with **cut point** λ

$$L_{\varphi,\lambda} = \{\omega \in \Sigma^* | \varphi(\omega) > \lambda\}$$

isolated by δ if

$$|\varphi(\omega) - \lambda| \geq \delta$$

LANGUAGES AND FORMAL SERIES

- (Rabin 63) 1pfa's with isolated cut point recognize the class of regular languages

- Languages recognized by MO-1qfa's, MM-1qfa's, 1qfc's with isolated cut point are regular
(for a survey, see Bertoni, Mereghetti, Palano, DLT2003)

THEOREM

- If φ describes the behaviour of a 1pfa (1qfa): $\operatorname{Im}(\varphi) \subseteq [0, 1]$

- $\varphi \in \mathbb{R}\langle\langle\Sigma^*\rangle\rangle$ is a **bounded series** if

$$\sup_{\omega\in\Sigma^*} |\varphi(\omega)| < \infty$$

- (π, μ, η) is a **bounded linear representation** if

$$\sup_{\omega\in\Sigma^*} \|\pi\mu(\omega)\|_2 < \infty$$

THEOREM

Theorem: if (π, μ, η) is a reduced (=of minimal dimension) linear representation of φ, then

$$(\pi, \mu, \eta) \text{ is bounded } \Leftrightarrow \varphi \text{ is bounded}$$

PROOF

$$(\pi, \mu, \eta) \text{ bounded} \Rightarrow \varphi \text{ bounded}$$

definition
of linear
representation

Schwarz
inequality

$$|\varphi(\omega)| = |\pi\mu(\omega)\eta^T| \leq \|\pi\mu(\omega)\|_2 \|\eta\|_2 < \infty$$

bounded by
hypotesis

PROOF

$$(\pi, \mu, \eta) \text{ bounded } \Leftarrow \varphi \text{ bounded}$$

(Schützenberger, 61) If (π, μ, η) is a reduced linear representation of φ, then there exist polynomials

$$P_1, \ldots, P_n, \quad Q_1, \ldots, Q_n$$

such that

$$\mu(\omega)_{ij} = \varphi(P_i \omega Q_j)$$

where, for $P \in \mathbb{R}\langle\Sigma^*\rangle$, $\varphi(P) = \sum_{\omega \in \Sigma^*} \varphi(\omega) P(\omega)$

PROOF

$$(\pi, \mu, \eta) \text{ bounded} \Leftarrow \varphi \text{ bounded}$$

$$Q_j = \sum_{m=1}^{\beta_j} d_{jm} v_{jm} \qquad P_i = \sum_{\ell=1}^{\alpha_i} c_{i\ell} u_{i\ell}$$

$$\mu(\omega)_{ij} = \varphi(P_i \omega Q_j) = \sum_{\substack{1 \leq \ell \leq \alpha_i, \\ 1 \leq m \leq \beta_j}} c_{i\ell} d_{jm} \varphi(u_{i\ell} \omega v_{jm})$$

PROOF

$$(\pi, \mu, \eta) \text{ bounded} \Leftarrow \varphi \text{ bounded}$$

$$Q_j = \sum_{m=1}^{\beta_j} d_{jm} v_{jm} \qquad P_i = \sum_{\ell=1}^{\alpha_i} c_{i\ell} u_{i\ell}$$

$$\mu(\omega)_{ij} = \varphi(P_i \omega Q_j) = \sum_{\substack{1 \leq \ell \leq \alpha_i, \\ 1 \leq m \leq \beta_j}} c_{i\ell} d_{jm} \varphi(u_{i\ell} \omega v_{jm})$$

$$\Downarrow$$

$$|\mu(\omega)_{ij}| \leq \sum_{\substack{1 \leq \ell \leq \alpha_i, \\ 1 \leq m \leq \beta_j}} | c_{i\ell}||d_{jm}||\varphi(u_{i\ell} \omega v_{jm})|$$

bounded by
hypotesis

THEOREM

(Bertoni, Mereghetti, Palano, DLT2003)
(π, μ, η) bounded linear representation of φ, and λ isolated for φ

$\Downarrow$

$L_{\varphi,\lambda}$ is regular

THEOREM

- (Bertoni, Mereghetti, Palano, DLT2003)
  (π, μ, η) bounded linear representation of φ, and λ isolated for φ

$$\Downarrow$$

$$L_{\varphi,\lambda} \text{ is regular}$$

- **Theorem:** $\varphi \in \mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ bounded and λ isolated for φ

$$\Downarrow$$

$$L_{\varphi,\lambda} \text{ is regular}$$

HADAMARD QUOTIENT OF RATIONAL POWER SERIES

HADAMARD OPERATIONS

- Hadamard product: $(\varphi \odot \psi)(\omega) = \varphi(\omega) \cdot \psi(\omega)$

- Hadamard quotient: $\frac{\varphi}{\psi}(\omega) = \frac{\varphi(\omega)}{\psi(\omega)}$

$$\text{when} \ \ \mathrm{supp}(\psi) = \Sigma^*$$

- $\mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ is closed under Hadamard product but not under Hadamard quotient

THEOREM

Theorem: $\xi = \frac{\varphi}{\psi}$ with $\varphi, \psi \in \mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$. If

(I) ξ is bounded
(II) ψ is bounded
(III) 0 is isolated for ψ

then λ isolated for $\xi \quad \Rightarrow \quad L_{\xi,\lambda}$ regular.

THEOREM

Theorem: $\xi = \frac{\varphi}{\psi}$ with $\varphi, \psi \in \mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$. If

(I) ξ is bounded
(II) ψ is bounded
(III) 0 is isolated for ψ

then λ isolated for ξ $\Rightarrow$ $L_{\xi,\lambda}$ regular.

Proof:

$$L_{\xi,\lambda} = \left\{\omega \in \Sigma^* \,\middle|\, \frac{\varphi(\omega)}{\psi(\omega)} > \lambda\right\} = \{\omega \in \Sigma^* \mid \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) > 0\}$$

THEOREM

Theorem: $\xi = \frac{\varphi}{\psi}$ with $\varphi, \psi \in \mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$. If

(I) ξ is bounded
(II) ψ is bounded
(III) 0 is isolated for ψ

then λ isolated for ξ $\Rightarrow$ $L_{\xi,\lambda}$ regular.

Proof:

$$L_{\xi,\lambda} = \left\{ \omega \in \Sigma^* \,\middle|\, \frac{\varphi(\omega)}{\psi(\omega)} > \lambda \right\} = \{\omega \in \Sigma^* \mid \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) > 0\}$$

we only have to show that 0 is isolated for $\varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega)$

PROOF

- We show that 0 is isolated for $\varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega)$

PROOF

- We show that 0 is isolated for $\varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega)$
- By hypothesis, λ is isolated for ξ, so $\exists\delta > 0$ such that either

$$\frac{\varphi(\omega)}{\psi(\omega)} \geq \lambda + \delta \text{ or } \frac{\varphi(\omega)}{\psi(\omega)} \leq \lambda - \delta$$

PROOF

- We show that 0 is isolated for $\varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega)$
- By hypothesis, λ is isolated for ξ, so $\exists\delta > 0$ such that either

$$\frac{\varphi(\omega)}{\psi(\omega)} \geq \lambda + \delta \quad \text{or} \quad \frac{\varphi(\omega)}{\psi(\omega)} \leq \lambda - \delta$$

$$\frac{\varphi(\omega)}{\psi(\omega)} \geq \lambda + \delta \;\Rightarrow\; \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) \geq \delta\psi^2(\omega)$$

$\geq \varepsilon > 0$

$$\Rightarrow\; \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) \geq \delta\varepsilon$$

PROOF

- We show that 0 is isolated for $\varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega)$
- By hypothesis, λ is isolated for ξ , so $\exists\delta > 0$ such that either

$$\frac{\varphi(\omega)}{\psi(\omega)} \geq \lambda + \delta \quad \text{or} \quad \frac{\varphi(\omega)}{\psi(\omega)} \leq \lambda - \delta$$

$$\frac{\varphi(\omega)}{\psi(\omega)} \geq \lambda + \delta \;\Rightarrow\; \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) \geq \delta\psi^2(\omega)$$

$\geq \varepsilon > 0$

$$\Rightarrow\; \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) \geq \delta\varepsilon$$

- By a symmetric reasoning

$$\frac{\varphi(\omega)}{\psi(\omega)} \leq \lambda - \delta \;\Rightarrow\; \varphi(\omega)\psi(\omega) - \lambda\psi^2(\omega) \leq -\delta\varepsilon$$

so $L_{\xi,\lambda}$ is regular

THEOREM

- Theorem: if one of conditions (I), (II), (III) is dropped, even if λ is isolated for ξ, $L_{\xi,\lambda}$ might not be context free.

THEOREM

- **Theorem**: if one of conditions (I), (II), (III) is dropped, even if λ is isolated for ξ, $L_{\xi,\lambda}$ might not be context free.

- **Lemma**: let $\varphi(a^n) = n^2 \sin^2(\pi n \theta)$, where $\theta = \frac{1+\sqrt{5}}{2}$,

  there exists a rational $\frac{\pi^2}{5} < \lambda < \frac{4\pi^2}{5}$ such that

  - λ is isolated for φ
  - $L_{\varphi,\lambda}$ is not context free

THEOREM

- **Theorem**: if one of conditions (I), (II), (III) is dropped, even if λ is isolated for ξ, $L_{\xi,\lambda}$ might not be context free.

- **Lemma**: let $\varphi(a^n) = n^2 \sin^2(\pi n \theta)$, where $\theta = \frac{1+\sqrt{5}}{2}$,

  there exists a rational $\frac{\pi^2}{5} < \lambda < \frac{4\pi^2}{5}$ such that

  - λ is isolated for φ
  - $L_{\varphi,\lambda}$ is not context free

- **Proof of the Theorem**: find 3 series which satisfy only 2 of the conditions and define the language $L_{\varphi,\lambda}$

PROOF

$$L_{\varphi,\lambda} = \{a^n \mid \varphi(a^n) = n^2 \sin^2(\pi n\theta) > \lambda\}$$

- Dropping condition (I): ξ not bounded

$$\xi_1(a^n) = \frac{\varphi_1(a^n)}{\psi_1(a^n)} = \frac{n^2 \sin^2(\pi n\theta)}{1} \qquad \lambda_1 = \lambda$$

- Dropping condition (II): ψ not bounded

$$\xi_2(a^n) = \frac{\varphi_2(a^n)}{\psi_2(a^n)} = \frac{n^2 \sin^2(\pi n\theta)+1}{n^2 \sin^2(\pi n\theta)+2} \qquad \lambda_2 = \frac{\lambda+1}{\lambda+2}$$

- Dropping condition (III): 0 not isolated for ψ

$$\xi_3(a^n) = \frac{\varphi_3(a^n)}{\psi_3(a^n)} = \frac{2^{-n}[n^2 \sin^2(\pi n\theta)+1]}{2^{-n}[n^2 \sin^2(\pi n\theta)+2]} \qquad \lambda_3 = \lambda_2$$

PROOF

$$L_{\varphi,\lambda} = \{a^n \mid \varphi(a^n) = n^2 \sin^2(\pi n\theta) > \lambda\}$$

Dropping condition (I): ξ not bounded

$$\xi_1(a^n) = \frac{\varphi_1(a^n)}{\psi_1(a^n)} = \frac{n^2 \sin^2(\pi n\theta)}{1} \qquad \lambda_1 = \lambda$$

Dropping condition (II): ψ not bounded

$$\xi_2(a^n) = \frac{\varphi_2(a^n)}{\psi_2(a^n)} = \frac{n^2 \sin^2(\pi n\theta)+1}{n^2 \sin^2(\pi n\theta)+2} \qquad \lambda_2 = \frac{\lambda+1}{\lambda+2}$$

Dropping condition (III): 0 not isolated for ψ

$$\xi_3(a^n) = \frac{\varphi_3(a^n)}{\psi_3(a^n)} = \frac{2^{-n}[n^2 \sin^2(\pi n\theta)+1]}{2^{-n}[n^2 \sin^2(\pi n\theta)+2]} \qquad \lambda_3 = \lambda_2$$

$$L_{\xi_1,\lambda_1} = L_{\xi_2,\lambda_2} = L_{\xi_3,\lambda_3} = L_{\varphi,\lambda}$$

HADAMARD QUOTIENT OF RATIONAL POWER SERIES AND TWO-WAY AUTOMATA

TWO WAY AUTOMATA

- 2pfa: $\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

TWO WAY AUTOMATA

2pfa: $\Xi = (\, Q, \pi, \{A^{-}(\sigma), A^{+}(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

$\{q_1, \ldots, q_m\}$

TWO WAY AUTOMATA

2pfa: $\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

$\{q_1, \ldots, q_m\}$

left and right endmarkers

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

  $\{q_1, \ldots, q_m\}$

  left and right endmarkers

- Input: $\#\omega\$$

$$\# \cdots \sigma \cdots \$$$

$$q_i \quad q_j$$

$$(A^+(\sigma))_{ij}$$

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

  $\{q_1, \ldots, q_m\}$

  left and right endmarkers

- Input: $\#\omega\$$

$$\# \cdots \sigma \cdots \$$$

$$q_j \quad q_i$$

$$(A^-(\sigma))_{ij}$$

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (Q, \pi, \{A^{-}(\sigma), A^{+}(\sigma)\}_{\sigma \in \Sigma \cup \{\#,\$\}}, \eta)$

  $\{q_1, \ldots, q_m\}$

  left and right endmarkers

- Input: $\#\omega\$$

$$\Big| \# \cdots \sigma \cdots \$$$

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (\underbrace{Q}_{\{q_1, \ldots, q_m\}}, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

  left and right endmarkers

- Input: $\#\omega\$$

$$\Big| \# \cdots \sigma \cdots \$$$

HALT

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (Q, \pi, \{A^{-}(\sigma), A^{+}(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

  $\{q_1, \ldots, q_m\}$

  left and right endmarkers

- Input: $\#\omega\$$

$$\Big|\# \cdots \sigma \cdots \$$$

HALT

accepts according to η

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#,\$\}}, \eta)$

  $\{q_1, \ldots, q_m\}$

  left and right endmarkers

- Input: $\#\omega\$$

  $$\Big| \# \cdots \sigma \cdots \$$$

  HALT

  accepts according to η

- **Event:** $p_\Xi : \Sigma^* \to [0,1]$

  is the probability of accepting the input word

TWO WAY AUTOMATA

- **2pfa:** $\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$

  $\{q_1, \ldots, q_m\}$

  left and right endmarkers

- Input: $\#\omega\$$

$$\#\cdots\sigma\cdots\$$$

HALT

accepts according to η

- **Event:** $p_\Xi : \Sigma^* \to [0,1]$

  is the probability of accepting the input word

- A 2pfa is **transient** if, for any input word, it halts with probability 1

THEOREM

(Anselmo, Bertoni, 93): Given a transient 2pfa

$$\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$$

the formal power series describing p_Ξ is the Hadamard quotient of two rational power series.

THEOREM

- (Anselmo, Bertoni, 93): Given a transient 2pfa

$$\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$$

the formal power series describing p_Ξ is the Hadamard quotient of two rational power series.

- Theorem: if all matrices $A^+(\sigma)$ are invertible, we give an explicit representation of p_Ξ in terms of Hadamard quotient.

THEOREM

- (Anselmo, Bertoni, 93): Given a transient 2pfa

$$\Xi = (Q, \pi, \{A^-(\sigma), A^+(\sigma)\}_{\sigma \in \Sigma \cup \{\#, \$\}}, \eta)$$

the formal power series describing p_Ξ is the Hadamard quotient of two rational power series.

- Theorem: if all matrices $A^+(\sigma)$ are invertible, we give an explicit representation of p_Ξ in terms of Hadamard quotient.

- Proof: without loss of generality

$$\pi = (\underset{q_1}{1}, 0, \ldots, 0) \qquad \eta = (0, \ldots, 0, \underset{q_m}{1})$$

PROOF

- For each word $\omega = \sigma_1 \cdots \sigma_n$, by letting $\# = \sigma_0$ and $\$ = \sigma_{n+1}$ we define

PROOF

- For each word $\omega = \sigma_1 \cdots \sigma_n$, by letting $\# = \sigma_0$ and $\$ = \sigma_{n+1}$ we define

$$M(w) = \begin{bmatrix} 0 & A_0^+ & 0 & & & 0 \\ A_1^- & 0 & A_1^+ & & & \\ 0 & A_2^- & \ddots & \ddots & & \\ & & \ddots & \ddots & A_n^+ & 0 \\ & & & A_{n+1}^- & 0 & A_{n+1}^+ \\ 0 & & & 0 & 0 & 0 \end{bmatrix}$$

$$A_i^+ = A^+(\sigma_i)$$

$$A_i^- = A^-(\sigma_i)$$

PROOF

- For each word $\omega = \sigma_1 \cdots \sigma_n$, by letting $\# = \sigma_0$ and $\$ = \sigma_{n+1}$ we define

$$
M(w) = \begin{array}{c}
\begin{matrix} \# & \sigma_1 & & \sigma_n & \$ & \text{HALT} \end{matrix} \\
\begin{bmatrix}
0 & A_0^+ & 0 & & & 0 \\
A_1^- & 0 & A_1^+ & & & \\
0 & A_2^- & \ddots & \ddots & & \\
 & & \ddots & \ddots & A_n^+ & 0 \\
 & & & A_{n+1}^- & 0 & A_{n+1}^+ \\
0 & & & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$A_i^+ = A^+(\sigma_i)$$

$$A_i^- = A^-(\sigma_i)$$

PROOF

- For each word $\omega = \sigma_1 \cdots \sigma_n$, by letting $\# = \sigma_0$ and $\$ = \sigma_{n+1}$ we define

$$M(w) = \begin{array}{c} \begin{array}{cccccc} \# & \sigma_1 & & \sigma_n & \$ & \text{HALT} \end{array} \\ \begin{bmatrix} 0 & A_0^+ & 0 & & & 0 \\ A_1^- & 0 & A_1^+ & & & \\ 0 & A_2^- & \ddots & \ddots & & \\ & & \ddots & \ddots & A_n^+ & 0 \\ & & & A_{n+1}^- & 0 & A_{n+1}^+ \\ 0 & & & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A_i^+ = A^+(\sigma_i)$$

$$A_i^- = A^-(\sigma_i)$$

$$\pi' = (\underbrace{e_1}_{\#}, \underbrace{\mathbf{0}}_{\sigma_1}, \ldots, \underbrace{\mathbf{0}}_{\sigma_n}, \underbrace{\mathbf{0}}_{\$}, \underbrace{\mathbf{0}}_{\text{HALT}}) \qquad e_1 = (1, 0, \ldots, 0)$$

$$\eta' = (\overbrace{\mathbf{0}}^{\#}, \overbrace{\mathbf{0}}^{\sigma_1}, \ldots, \overbrace{\mathbf{0}}^{\sigma_n}, \overbrace{\mathbf{0}}^{\$}, \overbrace{e_m}^{\text{HALT}})^T \qquad e_m = (0, \ldots, 0, 1)$$

PROOF

- For each word $\omega = \sigma_1 \cdots \sigma_n$, by letting $\# = \sigma_0$ and $\$ = \sigma_{n+1}$ we define

$$
M(w) = \begin{array}{c}
\begin{array}{cccccc} \;\#\; & \;\sigma_1\; & \quad & \;\sigma_n\; & \;\$\; & \text{HALT} \end{array} \\
\begin{bmatrix}
0 & A_0^+ & 0 & & & 0 \\
A_1^- & 0 & A_1^+ & & & \\
0 & A_2^- & \ddots & \ddots & & \\
 & & \ddots & \ddots & A_n^+ & 0 \\
 & & & A_{n+1}^- & 0 & A_{n+1}^+ \\
0 & & & 0 & 0 & 0
\end{bmatrix}
\end{array}
\qquad
\begin{array}{l} A_i^+ = A^+(\sigma_i) \\[2ex] A_i^- = A^-(\sigma_i) \end{array}
$$

$$
\pi' = (\underbrace{e_1}_{\#}, \underbrace{\mathbf{0}}_{\sigma_1}, \ldots, \underbrace{\mathbf{0}}_{\sigma_n}, \underbrace{\mathbf{0}}_{\$}, \underbrace{\mathbf{0}}_{\text{HALT}}) \qquad e_1 = (1, 0, \ldots, 0)
$$

$$
\eta' = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{0}, e_m)^T \qquad e_m = (0, \ldots, 0, 1)
$$

- $\pi'(M(\omega))^t \eta'$ is the probability of accepting ω in t steps

PROOF

$$p_{\Xi}(w) = \sum_{t=0}^{\infty} \pi'(M(w))^t \eta' = \pi' \left(\sum_{t=0}^{\infty} (M(w))^t \right) \eta'$$

PROOF

$$p_\Xi(w) = \sum_{t=0}^{\infty} \pi'(M(w))^t \eta' = \pi' \left(\sum_{t=0}^{\infty} (M(w))^t \right) \eta'$$

$$\Xi \ \text{transient} \qquad = \pi'(I - M(w))^{-1} \eta'$$

PROOF

$$p_\Xi(w) = \sum_{t=0}^{\infty} \pi'(M(w))^t \eta' = \pi' \left(\sum_{t=0}^{\infty} (M(w))^t \right) \eta'$$

Ξ transient $\qquad = \pi'(I - M(w))^{-1}\eta'$

$M(\omega) \in \mathbb{R}^{u \times u} \qquad = (I - M(w))^{-1}_{1,u}$

PROOF

$$p_\Xi(w) = \sum_{t=0}^{\infty} \pi'(M(w))^t \eta' = \pi' \left(\sum_{t=0}^{\infty} (M(w))^t \right) \eta'$$

Ξ transient

$$= \pi'(I - M(w))^{-1}\eta'$$

$M(\omega) \in \mathbb{R}^{u \times u}$

$$= (I - M(w))^{-1}_{1,u}$$

$$= \frac{(\mathrm{cof}\,(I - M(w)))_{u,1}}{\det(I - M(w))}$$

PROOF

$$p_\Xi(w) = \sum_{t=0}^{\infty} \pi'(M(w))^t \eta' = \pi' \left(\sum_{t=0}^{\infty} (M(w))^t \right) \eta'$$

$$\Xi \ \text{transient} \qquad = \pi'(I - M(w))^{-1} \eta'$$

$$M(\omega) \in \mathbb{R}^{u \times u} \qquad = (I - M(w))^{-1}_{1,u}$$

$$= \frac{(\mathrm{cof}\,(I - M(w)))_{u,1}}{\det(I - M(w))}$$

Idea: calculate p_Ξ as the quotient of determinants of fixed dimension matrices

TRIDIAGONAL BLOCK MATRICES

- (Molinari 08) Given the tridiagonal block matrix

$$M = \begin{bmatrix} A_1 & B_1 & & 0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ 0 & & C_{n-1} & A_n \end{bmatrix} \qquad \det B_i \neq 0$$

TRIDIAGONAL BLOCK MATRICES

- (Molinari 08) Given the tridiagonal block matrix

$$M = \begin{bmatrix} A_1 & B_1 & & 0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ 0 & & C_{n-1} & A_n \end{bmatrix} \qquad \det B_i \neq 0$$

- Its transfer matrix is

$$T = \begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_{n-1}^{-1} A_{n-1} & -B_{n-1}^{-1} C_{n-2} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1} A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix}$$

TRIDIAGONAL BLOCK MATRICES

- (Molinari 08) Given the tridiagonal block matrix

$$M = \begin{bmatrix} A_1 & B_1 & & 0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ 0 & & C_{n-1} & A_n \end{bmatrix} \qquad \det B_i \neq 0$$

- Its transfer matrix is

$$T = \begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_{n-1}^{-1} A_{n-1} & -B_{n-1}^{-1} C_{n-2} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1} A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix}$$

- Its determinant is

$$\det M = (-1)^{nm} \det[T_{11}] \det[B_1 \cdots B_{n-1}]$$

TRIDIAGONAL BLOCK MATRICES

- (Molinari 08) Given the tridiagonal block matrix with corners

$$M_z = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z}C_0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ zB_n & & C_{n-1} & A_n \end{bmatrix} \qquad \det B_i \neq 0$$

- Its transfer matrix is

$$T = \begin{bmatrix} -B_n^{-1}A_n & -B_n^{-1}C_{n-1} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1}C_0 \\ I_m & 0 \end{bmatrix}$$

- Its determinant is

$$\det M_z = \frac{(-1)^{nm}}{(-1)^m} \det[T - zI_{2m}] \det[B_1 \cdots B_n]$$

PROOF

$$p_\Xi(w) = \frac{(\mathrm{cof}\,(I - M(w)))_{u,1}}{\det(I - M(w))} = \frac{}{(-1)^m \det(T_{11}(w)) \cdot \det(A_0^+ \cdots A_{n+1}^+)}$$

transfer matrix of $I - M(\omega)$

$$I - M(w) = \begin{bmatrix} I_m & -A_0^+ & 0 & & & 0 \\ -A_1^- & I_m & -A_1^+ & & & \\ 0 & -A_2^- & \ddots & \ddots & & \\ & & \ddots & \ddots & -A_n^+ & 0 \\ & & & -A_{n+1}^- & I_m & -A_{n+1}^+ \\ 0 & & & 0 & 0 & I_m \end{bmatrix}$$

PROOF

$$p_{\Xi}(w) = \frac{\left(\mathrm{cof}\,(I - M(w))\right)_{u,1}}{\det(I - M(w))} = \frac{(-1)^m \det(Y_{11}(w) - Z_m) \cdot \det(A_0^+ \cdots A_{n+1}^+)}{(-1)^m \det(T_{11}(w)) \cdot \det(A_0^+ \cdots A_{n+1}^+)}$$

$$I - M(w) = \begin{array}{c} \\ \\ \\ \\ \\ Z_m \end{array} \begin{bmatrix} I_m & -A_0^+ & 0 & & & 0 \\ -A_1^- & I_m & -A_1^+ & & & \\ 0 & -A_2^- & \ddots & \ddots & & \\ & & \ddots & \ddots & -A_n^+ & 0 \\ & & & -A_{n+1}^- & I_m & -A_{n+1}^+ \\ \boxed{0} & & & 0 & 0 & I_m \end{bmatrix}$$

PROOF

$$p_{\Xi}(\omega) = \frac{(-1)^m \det(Y_{11}(w) - Z_m) \cdot \det(A_0^+ \cdots A_{n+1}^+)}{(-1)^m \det(T_{11}(w)) \cdot \det(A_0^+ \cdots A_{n+1}^+)}$$

PROOF

$$p_\Xi(\omega) = \frac{(-1)^m \det(Y_{11}(w) - Z_m) \cdot \det(A_0^+ \cdots A_{n+1}^+)}{(-1)^m \det(T_{11}(w)) \cdot \det(A_0^+ \cdots A_{n+1}^+)}$$

$$\sum_{w \in \Sigma^*} p_\Xi(w) w = \sum_{w \in \Sigma^*} \frac{\det(Y_{11}(w) - Z_m)}{\det(T_{11}(w))} w$$

PROOF

$$p_{\Xi}(\omega) = \frac{(-1)^m \det(Y_{11}(w) - Z_m) \cdot \det(A_0^+ \cdots A_{n+1}^+)}{(-1)^m \det(T_{11}(w)) \cdot \det(A_0^+ \cdots A_{n+1}^+)}$$

$$\sum_{w \in \Sigma^*} p_{\Xi}(w) w = \sum_{w \in \Sigma^*} \frac{\det(Y_{11}(w) - Z_m)}{\det(T_{11}(w))} w$$

$\dashv$ $\mathbb{R}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ is closed under sum and Hadamard product

OPEN PROBLEMS

- Study of the behaviour of 2qfa in relation to formal power series.

- Study of other classes of languages defined by formal power series and isolated cut point in relation to the Chomsky-Schützenberger classification.

THANK YOU