The Number of Convex Permutominoes⋆

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Abstract. Permutominoes are polyominoes defined by suitable pairs of permutations. In this paper we provide a formula to count the number of convex permutominoes of given perimeter. To this aim we define the transform of a generic pair of permutations, we characterize the transform of any pair defining a convex permutomino, and we solve the counting problem in the transformed space.

1 Introduction

A polyomino (also known as lattice animal) is a finite collection of square cells of equal size arranged with coincident sides. In this paper we consider a special class of polyominoes, namely the permutominoes, that we define in a purely geometric way. Actually, the term "permutomino" arises from the fact that this object can be defined by a diagram on the plane representing a pair of permutations. Such diagrams were introduced in [8] as a tool to study Schubert varieties and used in [6] (where the term "permutaomino" appeared for the first time) and [7] in relation to Kazhdan-Lusztig R-polynomials.

Counting the number of polyominoes and permutominoes is an interesting combinatorial problem, still open in its more general form; yet, for some subclasses of polyominoes, exact formulae are known. For instance, the number of convex polyominoes (i.e., whose intersection with any vertical or horizontal line is connected) of given perimeter has been obtained in [2], whereas the enumeration problem for some subclasses of convex permutominoes has been solved in [5]. In this paper, we provide an explicit formula for the number of convex permutominoes of a given perimeter.

Our counting technique is based on two basic facts. First, the boundary of every convex permutomino can be decomposed into four subpaths describing, in this order, a down/rightward, up/rightward, up/leftward, down/leftward stepwise movement. Second, for each abscissa (ordinate) there is exactly one vertical (horizontal) segment in the boundary with that coordinate. Actually, these two constraints hold not only for the boundary of convex permutominoes, but for a larger class of circuits we call admissible: in Section 3 we describe admissible circuits and we obtain their number $\mathscr{A}_n$ in Section 5. In Section 4 we characterize admissible circuits that do not define a permutomino: again we obtain their number $\mathscr{B}_n$ in Section 5. As a consequence, we get the number of convex permutominoes as the difference $\mathscr{A}_n - \mathscr{B}_n$.

2 Preliminaries

In this section, we shall recall some basic definitions and properties of polyominoes, permutominoes and generating functions.

2.1 Polyominoes and permutominoes

A *cell* is a closed subset of $\mathbf{R}^2$ of the form $[a,b] \times [a+1, b+1]$, where $a, b \in \mathbf{Z}$; we shall identify such a cell with the pair (a,b). Let us define a binary relation $\sim$ of *adjacency* between cells by letting $(a,b) \sim (a',b')$ if and only if $a = a'$ and $|b-b'| = 1$, or $|a-a'| = 1$ and $b = b'$. A subset P of $\mathbf{R}^2$ is a

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polyomino if and only if it is a finite nonempty union of cells that is connected by adjacency, i.e., such that if $(a,b),(a',b') \in P$ then there exist $(a_1,b_1),\dots,(a_k,b_k) \in P$ such that $(a,b) = (a_1,b_1) \sim (a_2,b_2) \sim \cdots \sim (a_k,b_k) = (a',b')$. See Figure 1 (a) for an example. A polyomino is defined up to translations; without loss of generality, we assume that the lowest leftmost vertex of the mininal bounding rectangle of the polyomino is placed at the point $(1,1)$.

Special types of polyominoes P are the following:

- P is *row-convex* if and only if $(a,b),(a',b) \in P$ and $a \le a'' \le a'$ imply $(a'',b) \in P$;
- P is *column-convex* if and only if $(a,b),(a,b') \in P$ and $b \le b'' \le b'$ imply $(a,b'') \in P$;
- P is *convex* if and only if it is both row- and column-convex;
- P is *directed* if and only if it contains at least one of the corner cells of its minimal bounding rectangle;
- P is *parallelogram* if and only if it is convex and contains at least a pair of opposite corner cells of its minimal bounding rectangle (e.g., both the lower-left and upper-right cells).

The (topological) border of a polyomino P is a disjoint union of simple closed curves; in particular, if there is only one curve, we say that P has *no holes*: all polyominoes in this work will have no holes. The border is a simple closed curve made of alternating vertical and horizontal nontrivial segments whose endpoints (*vertices*) have integral coordinates; conversely, every such a closed curve is the border of a polyomino without holes, so we shall freely identify polyominoes with their borders.

We say that P is a *permutomino of size n* if and only if its minimal bounding rectangle is a square of size $n-1$, and the border of P has exactly one vertical segment of abscissa z and one horizontal segment of ordinate z, for every $z \in \{1,\dots,n\}$. Notice that a convex permutomino of size n has perimeter $4(n-1)$.

In order to handle polyominoes we introduce the following definitions. A (stepwise) *simple path* is a sequence $P_1 = (x_1,y_1)$, $P'_1 = (x'_1,y'_1)$, $P_2 = (x_2,y_2)$, $P'_2 = (x'_2,y'_2)$ $\dots$, $P_m = (x_m,y_m)$, $P'_m = (x'_m,y'_m)$ of distinct points with integer coordinates such that, for all $i \in \{1,\dots,m\}$, $x_i = x'_i$, and $y'_i = y_{i+1}$ if $i < m$; notice that the segments $P_iP'_i$ are vertical, whereas the segments P'_iP_{i+1} are horizontal. More generaly, a *path* is a sequence of points P_1, P'_1, $\dots$, P_k, P'_k such that, for some $m \le k$, P_1, P'_1, $\dots$, P_m, P'_m is a simple path, and for all $i > m$, $P_i = P_{i-m}$ and $P'_i = P'_{i-m}$. A *circuit* is a simple path such that $y'_m = y_1$; when dealing with circuits, we shall implicitly assume that the subscripts are treated modulo m; so, for example P_{m+1} is just P_1. A point is a (self-)*crossing point* of a simple path if and only if it is the intersection of two segments, say $P_iP'_i$ and P'_jP_{j+1}; we also say that the path has a crossing at indices (i,j).

Clearly, visiting the border of a polyomino P counter-clockwise and starting from the highest vertex of the leftmost edge, we identify a circuit without crossing points: we call it the *boundary* of P and denote it by $P_1 = A, P'_1, P_2, P'_2, \dots, P_m, P'_m$ (see Figure 1 (a)). Notice that if P is a permutomino, then $m = n$.

In particular we consider four special points in the boundary of any polyomino P: let $A = P_1$ be the highest vertex of the leftmost edge, B be the leftmost vertex of the lowest edge, C be the lowest vertex of the rightmost edge, D be the rightmost of the highest edge (see Figure 1 (b)). Notice that, if P is convex, then the subsequence of vertices between A and B (B and C, C and D, D and P'_m, respectively) is a path directed down/rightward (up/rightward, up/leftward, down/leftward, respectively); see Figure 1 (c).

2.2 Generating functions

The *generating function* $f(z)$ of the sequence $\{a_n\}_n$ is defined as [9] $f(z) = \sum_n a_n z^n$; it is well-known that

$$z f'(z) = \sum_n n a_n z^n \qquad \text{and} \qquad f(z) \cdot g(z) = \sum_n \left(\sum_{k=0}^{n} a_n b_{n-k} \right) z^n,$$

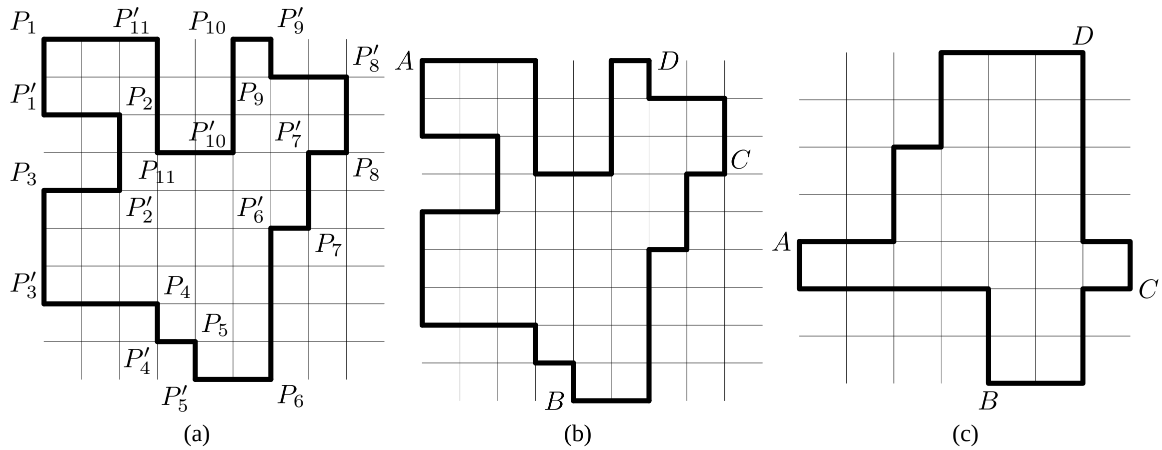

Fig. 1. (a) The boundary of a polyomino. (b) The extreme points of a polyomino. (c) The extreme points of a convex polyomino.

where $g(z) = \sum_n b_n z^n$. Some examples of generating functions that we will need in the following are:

$$\frac{1}{\sqrt{1-4z}} = \sum_n \binom{2n}{n} z^n$$
$$\frac{1}{1-4z} = \sum_n 4^n z^n.$$

As a consequence of the previous facts, we have that

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n. \tag{1}$$

3 Permutominoes, permutations, and transform

In this section we illustrate the relationship between the set of permutominoes of size n and the set

$$\Pi_n = \{(\sigma,\tau) \mid \sigma,\tau \in S(n), \sigma(x) \neq \tau(x) \text{ for every } x, \text{ and } \sigma(1) > \tau(1)\}.$$

Consider a permutomino P of size n and let $P_1, P'_1, P_2, P'_2, \ldots, P_n, P'_n$ be its boundary. By definition, for any $z \in \{1,\ldots,n\}$ there is exactly one index i such that P_i and P'_i have abscissa z and there is exactly one index j such that P'_j and P_{j+1} have ordinate z. Thus, a permutomino P of size n uniquely determines a pair of permutations $(\sigma,\tau) \in \Pi_n$ (which we call the *permutation pair* of P): $\sigma(x)$ and $\tau(x)$ are defined as the respective ordinates of the (unique) points P_i and P'_i with abscissa x. In particular, observe that $A = P_1 = (1,\sigma(1))$, $B = (\tau^{-1}(1),1)$, $C = (n,\sigma(n))$, $D = (\tau^{-1}(n),n)$.

Conversely, a pair $(\sigma,\tau) \in \Pi_n$ does not always define a permutomino. However, one can always consider the set of points ($i \in \{1,\ldots,n\}$)

$$T_i = (x_i, \tau(x_i)) \quad \text{and} \quad S_i = (x_i, \sigma(x_i)),$$

where, for every $i < n$,

$$x_1 = 1, \quad x_i = \sigma^{-1}(\tau(x_{i-1})),$$

We define the *path of* (σ,τ) as the path $S_1, T_1, S_2, T_2, \ldots, S_n, T_n$. Notice that this path needs not be simple, as Figure 2 (c) illustrates. However, if the $2n$ points are all distinct, the path is indeed a circuit with exactly one vertical (and horizontal) segment for every abscissa (and ordinate), see Figure 2 (a); yet, the circuit may contain crossing points (see Figure 2 (b)).

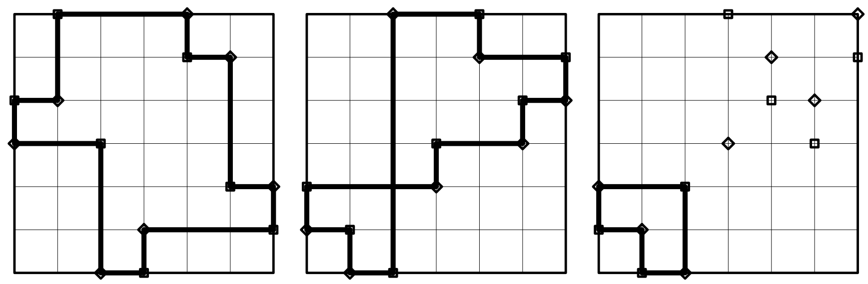

Fig. 2. (a) The permutomino of size $n = 7$ defined by $\sigma = (5,7,4,1,6,3,2)$ and $\tau = (4,5,1,2,7,6,3)$ (squares represent σ, whereas lozenges represent τ). (b) The path of $\sigma = (3,2,1,4,7,5,6)$ and $\tau = (2,1,7,3,6,4,5)$, which does not define a permutomino. (c) The path of $\sigma = (3,2,1,4,6,5,7)$ and $\tau = (2,1,3,7,5,4,6)$, which does not define a permutomino of size $n = 7$.

Remark 1. A pair of permutations in Π_n is the permutation pair of a permutomino P if and only if its path has exactly $2n$ distinct vertices and has no crossing points. In this case its path coincides with the boundary of P, that is $S_i = P_i$ and $T_i = P_i'$ for every i.

We notice that this remark is actually the definition of permutomino as introduced in [6]. Indeed, our definition of path of a pair of permutation recalls the geometric contruction used in [6], even though there are some differences (for instance in the case of Figure 2 (c)).

We now introduce a map $F_n : \Pi_n \to \Phi_n$ where Φ_n is the set of pairs of endofunctions of $\{1,\dots,n\}$. For any pair $(\sigma,\tau) \in \Pi_n$ we set $F_n(\sigma,\tau) = (v,h)$ where

$$v(1) = 1, \qquad v(i+1) = \sigma^{-1}(\tau(v(i)))$$

$$h(1) = \tau(1), \qquad h(i+1) = \tau(\sigma^{-1}(h(i)))$$

for every $i \in \{1,\dots,n-1\}$. The pair (v,h) is called the *transform* of (σ,τ); Figure 3 shows an example of permutomino and the transform (v,h) of its permutation path (σ,τ). The transform F_n has the following geometric interpretation: $v(i)$ is the abscissa of the i-th vertical edge along the path of (σ,τ), whereas $h(i)$ is the ordinate of the i-th horizontal edge along the same path. Indeed, the following proposition holds:

Proposition 1. *Let (σ,τ) be a pair of permutations, $S_1,T_1,S_2,T_2,\dots,S_n,T_n$ be its path and $(v,h) = F_n(\sigma,\tau)$ be its transform. Then one has*

$$S_i = (v(i),h(i-1)) \quad \text{and} \quad T_i = (v(i),h(i)) \tag{2}$$

for every $i \in \{1,\dots,n\}$, where we let $h(0) = h(n)$ for the sake of simplicity.

Notice that the path goes rightwards (resp. leftwards) according to whether v is increasing or decreasing and goes upwards (resp. downwards) according to whether h is increasing or decreasing (see Figure 3 for an example). Also observe that the functions v and h need not to be permutations; for instance this is the case for the permutation pair of Figure 2 (c).

The transform of the permutation pair of a convex permutomino has special properties, that can be observed in Figure 3 (right). To illustrate them, we introduce the following definition.

Definition 1. *The pair $(v,h) \in \Phi_n$ is said to be* admissible *whenever $(v,h) \in S(n) \times S(n)$ and, setting $v_* = v^{-1}(1)$, $h_* = h^{-1}(1)$, $v^* = v^{-1}(n)$, and $h^* = h^{-1}(n)$, one has*

– $1 = v_* \le h_* < v^* \le h^*$,

– *v is increasing in $\{1,\dots,v^*\}$ and decreasing in $\{v^*,\dots,n\}$,*
– *h is decreasing in $\{1,\dots,h_*\}$, increasing in $\{h_*,\dots,h^*\}$, decreasing in $\{h^*,\dots,n\}$, with $h(n)>h(1)$.*

The set of admissible pairs in Φ_n shall be denoted by $\Phi_n^{\mathscr{A}}$.

This definition is justified by the following fact:

Proposition 2. *Let $(\sigma,\tau)\in\Pi_n$ be a pair of permutations. Then $F_n(\sigma,\tau)$ is admissible if and only if the path of (σ,τ) is a circuit that can be decomposed into four subpaths directed, in this order, down/rightward, up/rightward, up/leftward, down/leftward.*

The previous proposition leads us to define *admissible* a circuit that can be decomposed as in the statement (see, for example, Figure 3 and 4), and to introduce the set

$$\Pi_n^{\mathscr{A}}=\{(\sigma,\tau)\in\Pi_n \mid \text{the path of } (\sigma,\tau) \text{ is an admissible circuit}\}.$$

Indeed, the previous proposition can be extended as follows:

Proposition 3. *The sets $\Pi_n^{\mathscr{A}}$ and $\Phi_n^{\mathscr{A}}$ are in bijection via F_n.*

Proof. Proposition 2 implies that $F_n(\Pi_n^{\mathscr{A}})\subseteq\Phi_n^{\mathscr{A}}$; we need to prove bijectivity. Given two permutations (v,h) with $v(1)=1$, set

$$\sigma(x)=h(v^{-1}(x)-1) \quad \text{and} \quad \tau(x)=h(v^{-1}(x)).$$

Letting $F_n(\sigma,\tau)=(v',h')$, one can easily verify that $v(i)=v'(i)$ and $h(i)=h'(i)$ by induction on i. For uniqueness, it is sufficient to use the definition of F_n and Proposition 1. □

The preimage $(\sigma,\tau)\in\Pi_n^{\mathscr{A}}$ of a $(v,h)\in\Phi_n^{\mathscr{A}}$ is called the *antitransform* of (v,h), and the path of (σ,τ) is called the *anticircuit* of (v,h).

In particular, since convex permutominoes are admissible circuits, we obtain:

Corollary 1. *If (σ,τ) is the permutation pair of a convex permutomino P, then its transform (v,h) is admissible. Moreover, $A=S_{v_*}$, $B=T_{h_*}$, $C=S_{v^*}$ and $D=T_{h^*}$.*

Observe however that, in general, the converse of the previous corollary is not true, because an admissible circuit may contain crossing points, as shown in Figure 4 (Left).

4 Crossing points

At this point, it should be clear that, if $(\sigma,\tau)\in\Pi_n^{\mathscr{A}}$, then either its path is a permutomino or it has crossing points. Such points can ensue only from one of the following two cases: the up/rightward subpath intersects the down/leftward subpath (crossing point of the *first type*, as in Figure 4), or the down/rightward subpath intersects the up/leftward subpath (crossing point of the *second type*, as in Figure 5). Actually, we will show that the crossing points do satisfy stronger conditions.

Lemma 1. *Let $(v,h)\in\Phi_n^{\mathscr{A}}$ and $\mathcal{P}$ be its anticircuit. Then, the crossing points of $\mathcal{P}$ (if any) are all of the same type.*

Proof. Let X be a crossing point of first type. Then the down/rightward subpath of $\mathcal{P}$ is all included in the square with vertices $(1,1)$ and X; analogously, the up/leftward subpath of $\mathcal{P}$ is all included in the square with vertices X and (n,n). This implies that these subpaths never cross each other, so any other crossing point must be of the first type. □

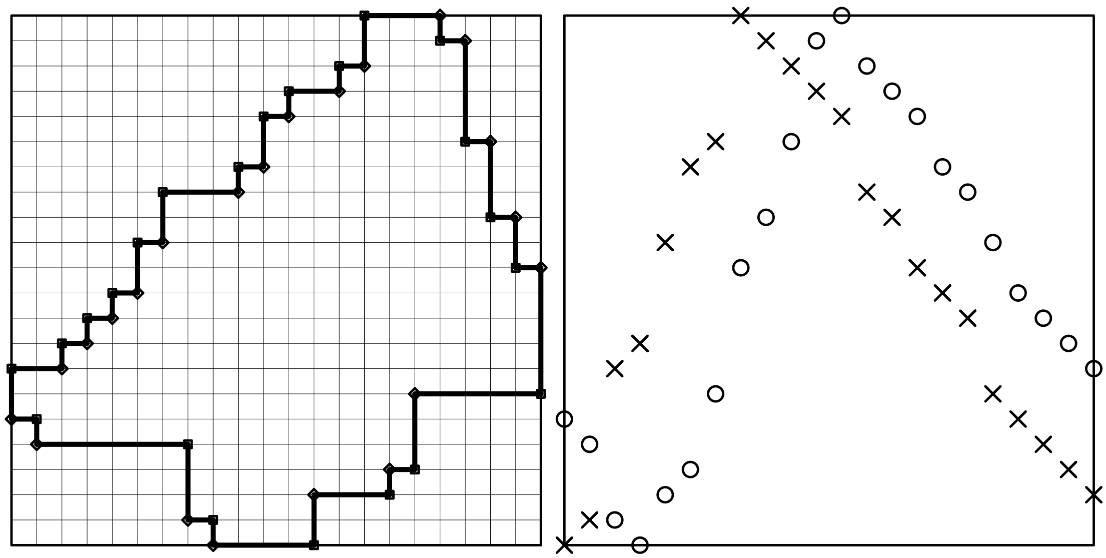

Fig. 3. (Left) A convex permutomino P. (Right) The diagram of the transform (v,h) (circles represent h, whereas crosses represent v) of the permutation pair of P.

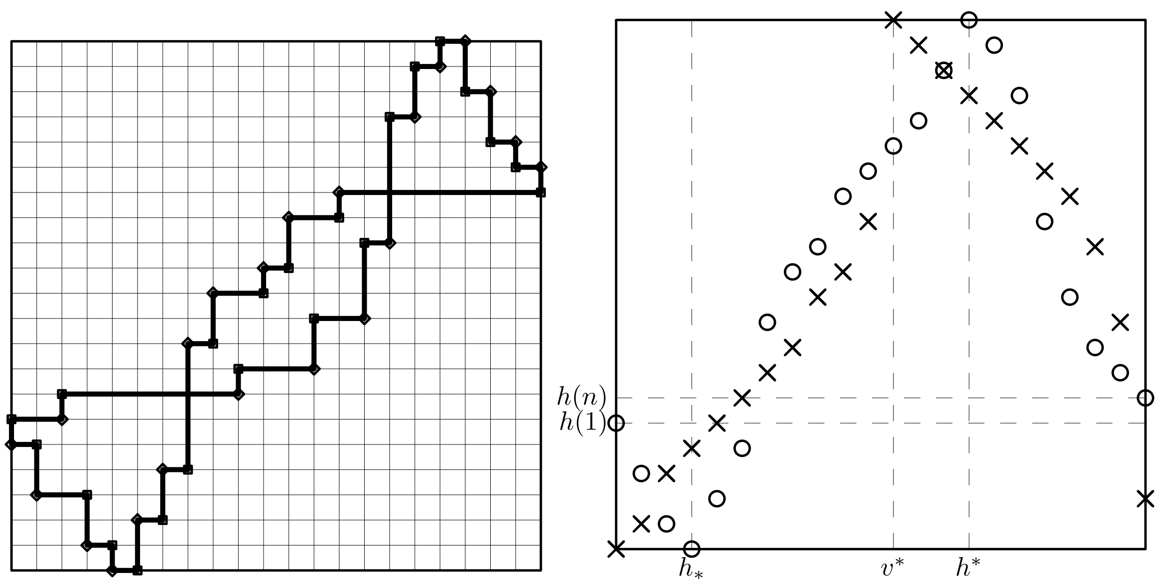

Fig. 4. (Left) An admissible circuit which is not a permutomino. (Right) The diagram of the pair of functions (v,h) corresponding to the circuit (circles represent h, whereas crosses represent v).

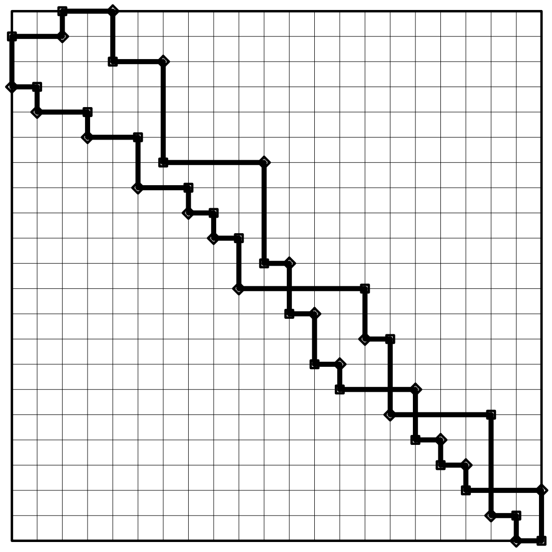

Fig. 5. The anticircuit of (v,h), which is not a permutomino and has crossing points of the second type.

Now consider the sequence of crossing points of $\mathcal{P}$, ordered so that their abscissas are (strictly) increasing. Clearly, the circuit $\mathcal{P}$ passes through all these points once in this order and then again in the reverse order. Notice that also the ordinates turns out to be ordered: if the crossing points are of the first (resp. second) type, then they are strictly increasing (resp. decreasing).

For the sake of simplicity, thanks to Lemma 1, we now focus on admissible pairs whose anticircuit $\mathcal{P}$ has only crossing points (if any) of the first type. The other case can be dealt with by simmetry. Under this hypothesis, the crossing points of $\mathcal{P}$ can be classified into two groups, as illustrated in Figure 6.

- A crossing point X is *UL* if it is the intersection of an *upward* segment $S_\mu T_\mu$(where $h_* < \mu < v^*$) with a *leftward* segment $T_\lambda S_{\lambda+1}$(where $\lambda > h^*$); in this case, $X = (v(\mu), h(\lambda))$ by Proposition 1.
- A crossing point X is *RD* if it is the intersection of a *righward* segment $T_\rho S_{\rho+1}$ (where $h_* < \rho < v^*$) with a *downward* segment $S_\delta T_\delta$ (where $\delta > h^*$); in this case $X = (v(\delta), h(\rho))$ by Proposition 1.

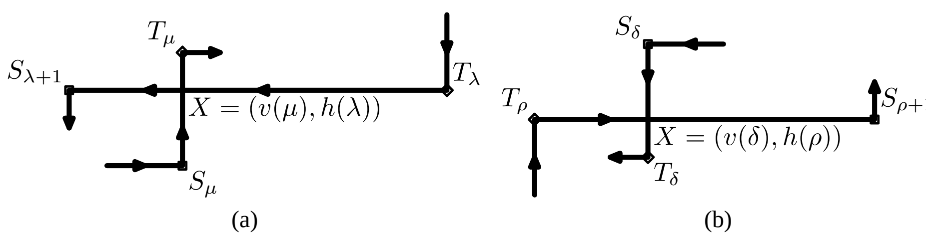

Fig. 6. (a) A UL crossing point at indices (μ,λ); (b) A RD crossing point at indices (δ,ρ).

It is easy to see that the first crossing point of $\mathcal{P}$ is UL, the last one is RD, whereas the inner ones alternate. In particular this implies that the number of crossing points is always even. Thus, letting $X_1, X_2, \ldots, X_{2k-1}, X_{2k}$ be the ordered sequence of crossing points of $\mathcal{P}$, there exists a sequence of indices

$$\mu_1 \le \rho_1 < \mu_2 \le \rho_2 < \cdots < \mu_k \le \rho_k < \delta_k \le \lambda_k < \delta_{k-1} \le \cdots < \delta_1 \le \lambda_1$$

such that, for every $i = 1, \ldots, k$,

$$X_{2i-1} = (v(\mu_i), h(\lambda_i)) \quad \text{and} \quad X_{2i} = (v(\delta_i), h(\rho_i));$$

note that the crossing points with odd indices are UL whereas those with even indices are RD. Since the abscissa —and the ordinates— are increasing, we also have

$$v(\mu_1) < v(\delta_1) < v(\mu_2) < \cdots < v(\mu_k) < v(\delta_k)$$

and

$$h(\lambda_1) < h(\rho_1) < h(\lambda_2) < \cdots < h(\lambda_k) < h(\rho_k).$$

Actually, the points X_j's all lay on the diagonal with endpoints $(1,1)$ and (n,n). (On the other hand, if the crossing points are of the second type, it turns out that they all lay on the diagonal with endpoints $(1,n)$ and $(n,1)$.) Indeed, we show that $v(\mu_i) = h(\lambda_i)$ and $v(\delta_i) = h(\rho_i)$ for every i, that is, the previous chains of inequalities coincide. Consider any UL crossing point $X_{2i-1} = (v(\mu_i), h(\lambda_i))$, and the new circuits (that, in general, may themselves contain crossing points):

$$\begin{gathered} S_1, T_1, \ldots, S_{h_*}, T_{h_*}, \ldots, S_{\mu_i}, X_{2i-1}, S_{\lambda_i+1}, T_{\lambda_i+1}, \ldots, S_n, T_n \\ X_{2i-1}, T_{\mu_i}, S_{\mu_i+1}, T_{\mu_i+1} \ldots, S_{v^*}, T_{v^*}, \ldots, S_{h^*}, T_{h^*}, \ldots, S_{\lambda_i} T_{\lambda_i}. \end{gathered}$$

Observe that, by Corollary 1, the first circuit contains A and B and is included in the square with vertices $(1,1)$ and X_{2i-1}, whereas the second circuit contains C and D and is included in the square with vertices X_{2i-1} and (n,n). Also, by Proposition 1, the second circuit is the anticircuit of the restrictions of v and h to the set $\{\mu_i, \ldots, \lambda_i\}$ (up to suitable translations). Hence, such restrictions are bijections onto the sets $\{v(\mu_i), \ldots, n\}$ and $\{h(\lambda_i), \ldots, n\}$, respectively, and hence one gets $v(\mu_i) = h(\lambda_i)$. Similarly, any RD crossing point $X_{2i} = (v(\delta_i), h(\rho_i))$ splits the circuit $\mathcal{P}$ into two circuits: the one included in the square with endpoints $(1,1)$ and X_{2i}, and the other included in the square with endpoints X_{2i} and (n,n). Thus, one obtains $v(\delta_i) = h(\rho_i)$ for every $i = 1, \ldots, k$.

Hence, the crossing points split the circuit $\mathcal{P}$ into $2k+1$ new circuits $\mathcal{P}_j$, for $j = 0, \ldots, 2k$ (see Figure 7). Each of them has no crossing point, thus it is the boundary of a convex polyomino. Actually, reasoning as above, one can prove that each $\mathcal{P}_j$ is the boundary of the permutomino whose permutation pair (σ_j, τ_j) is defined as follows (setting $\delta_0 = 1$, $\mu_{k+1} = v^*$, and up to suitable traslations of domains and codomains):

- σ_{2i} is the restriction of σ to the domain $\{v(\delta_i), \ldots, v(\mu_{i+1})\}$ for every $i = 0, 1, \ldots, k$;
- τ_{2i} is the restriction of τ to the domain $\{v(\delta_i), \ldots, v(\mu_{i+1})\}$, except for $\tau_0(v(\mu_1)) = v(\mu_1)$, $\tau_{2k}(v(\delta_k)) = v(\delta_k)$, $\tau_{2i}(v(\delta_i)) = v(\delta_i)$ for every $i = 1, 2, \ldots, k$, and $\tau_{2i}(v(\mu_{i+1})) = v(\mu_{i+1})$, for every $i = 0, 1, \ldots, k-1$;
- σ_{2i-1} is the restriction of σ to the domain $\{v(\mu_i), \ldots, v(\delta_i)\}$ for every $i = 1, 2 \ldots, k$, except for $\sigma_{2i-1}(v(\mu_i)) = v(\mu_i)$ and $\sigma_{2i-1}(v(\delta_i)) = v(\delta_i)$;
- τ_{2i-1} is the restriction of τ to the domain $\{v(\mu_i), \ldots, v(\delta_i)\}$ for every $i = 1, 2 \ldots, k$.

Intuitively, the pairs (σ_j, τ_j) are the restrictions of (σ, τ) to suitable subintervals of $\{1, \ldots, n\}$, except for the interval endpoints in correspondence with crossing points. Notice that the permutominoes with boundaries $\mathcal{P}_0$ and $\mathcal{P}_{2k}$ are both directed-convex, while the other ones are parallelograms. So, we have proved the following theorem.

Theorem 1. *Let $(v,h) \in \Phi_n^{\mathcal{A}}$ and $\mathcal{P}$ be its anticircuit. Then, either $\mathcal{P}$ is the boundary of a convex permutomino, or $\mathcal{P}$ has an even number $2k$ of crossing points of the same type. In the latter case, either all crossing points lay on the diagonal with endpoint $(1,1)$ and (n,n), or they all lay on the diagonal with endpoints $(1,n)$ and $(n,1)$. Moreover, $\mathcal{P}$ determines $2k+1$ new circuits, each of which is the boundary of a convex permutomino: the $2k-1$ inner permutominoes are parallelogram, whereas the two outer ones are directed-convex.*

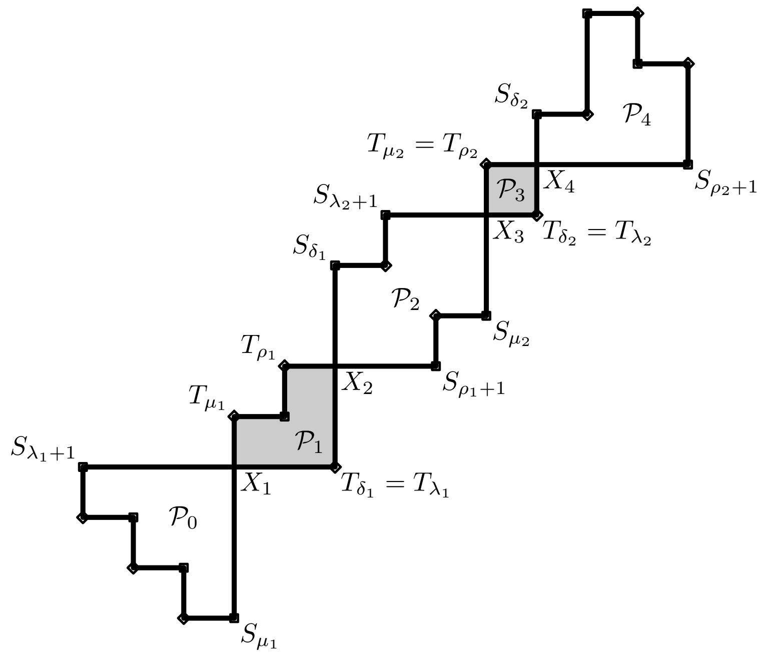

Fig. 7. How the crossing points split a circuit into a sequence of permutominoes.

5 Counting convex permutominoes

Theorem 1 provides a precise characterization of the admissible pairs having crossing points. We will call them *bad* pairs, since they do not define a permutomino. Hence, in order to count the number $\mathscr{C}_n$ of convex permutominoes of size n, we first obtain the number $\mathscr{A}_n$ of admissible pairs and then the number $\mathscr{B}_n$ of the bad ones. Our main result is hence given by a subtraction $\mathscr{C}_n = \mathscr{A}_n - \mathscr{B}_n$.

To this aim, we recall that in [5] the authors succeeded in giving an explicit formula for counting the number of some subclasses of convex permutominoes; more precisely, they proved that the number d_n of directed-convex permutominoes with size n is

$$d_n = \frac{1}{2}\binom{2(n-1)}{n-1}, \tag{3}$$

whereas the number of parallelogram permutominoes of size n equals the $(n-1)$-th Catalan number

$$p_n = c_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}. \tag{4}$$

We start by computing the number of admissible pairs:

Theorem 2. *For every $n \geq 2$, the number of admissible pairs is*

$$\mathscr{A}_n = \sum_{s=0}^{n-2}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{n-2}{t}\binom{n-2}{u+s-t}.$$

Proof. By the definition of admissible pair, we first have to choose the values h_*, v^* and h^* such that $1 \leq h_* < v^* \leq h^* \leq n$. Once these values are fixed, take any two subsets of $\{2,\ldots,n-1\}$, say V and H, with cardinalities v^*-2 and h^*-h_*-1, respectively. Let now v be the unique permutation such that $v(1)=1$, $v(v^*)=n$, $v(\{2,\dots,v^*-1\})=V$, with v increasing in such an interval, and decreasing in the remaining interval $\{v^*+1,\dots,n\}$; similarly, let h be the unique permutation such that $h(h_*)=1$, $h(h^*)=n$, $h(\{h_*+1,\dots,h^*-1\})=H$, with h increasing in such an interval, and decreasing (cyclically) in the remaining interval $\{h^*+1,\dots,n\}\cup\{1,\dots,h_*-1\}$. This is clearly an admissible pair, and it is uniquely determined by the choice of V and H. So, the number of admissible pairs is

$$\mathscr{A}_n=\sum_{1\le h_*<v^*\le h^*\le n}\binom{n-2}{v^*-2}\binom{n-2}{h^*-h_*-1}.$$

Substituting $s=h^*-2$, $t=v^*-2$ and $u=v^*-h_*-1$ in the previous summation, we obtain the result. □

As proved in a separate work [1], the previous summation turns out to be equal to

$$2n4^{n-3}-(n-2)\binom{2(n-2)}{n-2}. \tag{5}$$

As shown in the previous section, bad admissible pairs can be depicted as particular sequences of parallelogram and direct convex permutominoes. We proceed with two counting lemmata that will lead to an explicit formula for $\mathscr{B}_n$ in Theorem 3.

Lemma 2. *For every $m\ge 0$, we have*

$$\sum_k \sum_{\substack{t_1,\dots,t_{2k-1}>0\\ t_1+\cdots+t_{2k-1}=m+1}} c_{t_1}\cdots c_{t_{2k-1}}=\binom{2m}{m}.$$

Proof. The Catalan number c_t counts the number of trees[1] with $2t$ edges whose internal nodes have exactly two children. So, the left-hand side of the formula counts the number of ordered forests made by an odd number of trees, each being non-trivial and with all internal nodes having two children exactly, where the overall number of edges is $2m+2$.

The set $\mathcal{F}_{2m+2}$ of such forests is in bijection with the set $\mathcal{T}_{2m+2}$ of the trees with $2m+2$ edges, all internal nodes with exactly two children, and the root with an even number of children whose half is odd. Indeed, let $T_1,\dots,T_{2k-1}$ be any forest in $\mathcal{F}_{2m+2}$ and consider, for every i, the two subtrees T_i' and T_i'' rooted at the two children of the root of T_i. The corresponding tree in $\mathcal{T}_{2m+2}$ is obtained by attaching $T_1',T_1'',\dots,T_{2k-1}',T_{2k-1}''$ at a new root. Conversely, every tree in $\mathcal{T}_{2m+2}$ can be obtained from a suitable forest in $\mathcal{F}_{2m+2}$. Thus, the result is proved if we show that the cardinality of $\mathcal{T}_{2m+2}$ is exactly $\binom{2m}{m}$.

This follows from the general formula of [3], with $R=\{2,6,10,14,\dots\}$, $N=\{2\}$ and $L=\{1\}$, that yields the generating function $T(z)=1+z^2/\sqrt{1-4z^2}$ of the sequence $\{|\mathcal{T}_m|\}$. Since $G(z)=1/\sqrt{1-4z}$ is the generating function for the central binomial coefficient $\binom{2m}{m}$, we obtain that

$$T(z)=1+z^2G(z^2)=1+\sum_{m\ge 0}\binom{2m}{m}z^{2m+2}.\square$$

Lemma 3. *For every $m\ge 0$, we have*

$$\sum_{s=0}^{m}4^s\binom{2(m-s)}{m-s}=\binom{2m}{m}(2m+1).$$

[1] Here and henceforth, by *tree* we mean ordered rooted tree.

Proof. We prove that the generating function for the left-hand side is the same as the one for the right-hand side. The left-hand side is a convolution, whose generating function is the product of $1/(1-4z)$ and $1/\sqrt{1-4z}$, i.e., $(1-4z)^{-3/2}$. For the right-hand side, notice that

$$\sum_{m=0}^{\infty}(2m+1)\binom{2m}{m}z^m = 2\sum_{m=0}^{\infty} m\binom{2m}{m}z^m + \sum_{m=0}^{\infty}\binom{2m}{m}z^m =$$
$$= 2z\frac{d}{dz}\left(\frac{1}{\sqrt{1-4z}}\right) + \frac{1}{\sqrt{1-4z}} = (1-4z)^{-3/2}. \square$$

Theorem 3. *For every $n \geq 3$, the number of admissible pairs that do not define a permutomino is*

$$\mathscr{B}_n = (n-1)\binom{2(n-2)}{n-2} - 4^{n-2}.$$

Proof. By Theorem 1, if an admissible pair does not define a permutomino, then it defines a sequence of $2k+1$ permutominoes $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{2k}$ where $\mathcal{P}_0$ and $\mathcal{P}_{2k}$ are direct-convex and $\mathcal{P}_1, \ldots, \mathcal{P}_{2k-1}$ are parallelogram. Letting n_i be the size of $\mathcal{P}_i$, we have $\sum_i n_i = n + 2k$, since each crossing point X_i ($i \in \{1, \ldots, 2k\}$) coincides with both the upper-rightmost corner of $\mathcal{P}_{i-1}$ and lower-leftmost corner of $\mathcal{P}_i$. Hence the number of admissible pairs that do not define a permutomino are

$$\mathscr{B}_n = 2 \cdot \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\substack{n_0, \ldots, n_{2k} \geq 2 \\ n_0 + \cdots + n_{2k} = n+2k}} d_{n_0} p_{n_1} \cdots p_{n_{2k-1}} d_{n_{2k}}$$

where d_n and p_n are the numbers of direct-convex and parallelogram permutominoes of size n, respectively. The factor 2 accounts for the symmetry between crossing points of the first and second type. Now, set $r = n_0 - 1$, $s = r - 1 + n_{2k}$ and $t_i = n_i - 1$ for every $i \in \{1, \ldots, 2k-1\}$. Recalling Equations (3) and (4), we have

$$\mathscr{B}_n = 2 \cdot \sum_{s=2}^{n-2} \left(\sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\substack{t_1, \ldots, t_{2k-1} \geq 1 \\ t_1 + \cdots + t_{2k-1} = n-1-s}} c_{t_1} \cdots c_{t_{2k-1}} \right) \left(\frac{1}{4} \sum_{r=1}^{s-1} \binom{2r}{r} \binom{2(s-r)}{s-r} \right).$$

Applying Lemma 2 with $m = n-2-s$ and Equation (1) with $n = s$, we obtain

$$\mathscr{B}_n = \frac{1}{2} \cdot \sum_{s=2}^{n-2} \binom{2(n-s-2)}{n-s-2} \left(4^s - 2\binom{2s}{s}\right) =$$
$$= \frac{1}{2} \cdot \sum_{s=0}^{n-2} \binom{2(n-s-2)}{n-s-2} \left(4^s - 2\binom{2s}{s}\right) + \frac{1}{2}\binom{2(n-2)}{n-2} =$$
$$= \frac{1}{2} \cdot \sum_{s=0}^{n-2} \binom{2(n-s-2)}{n-s-2} 4^s - \sum_{s=0}^{n-2} \binom{2(n-s-2)}{n-s-2} \binom{2s}{s} + \frac{1}{2}\binom{2(n-2)}{n-2}.$$

Now, applying Lemma 3 with $m = n-2$ to the first summand, and using again Equation (1), we obtain the result. $\square$

From Theorems 2 and 3 and equation (5), we are now able to show the main result.

Corollary 2. *The number of convex permutominoes of size $n \geq 2$ is*

$$\mathscr{C}_n = \mathscr{A}_n - \mathscr{B}_n = 2(n+2)4^{n-3} - (2n-3)\binom{2(n-2)}{n-2}$$

The first few terms of the sequences $\mathscr{A}_n$, $\mathscr{B}_n$ and $\mathscr{C}_n$ are given in Table 1.

Table 1. The number $\mathscr{A}_n$ of admissible pairs, the number $\mathscr{B}_n$ of admissible pairs with crossings and the number $\mathscr{C}_n$ of convex permutominoes of size n.

n	2	3	4	5	6	7	8	9	10
$\mathscr{A}_n$	1	4	20	100	488	2324	10840	49704	224720
$\mathscr{B}_n$	0	0	2	16	94	488	2372	11072	50294
$\mathscr{C}_n$	1	4	18	84	394	1836	8468	38632	174426

6 Conclusions

In this paper we have presented a novel technique to study permutominoes: we defined the transform of a pair of permutations that can be thought of as a sort of duality. More precisely, even though the set of pairs of permutations (σ, τ) defining a convex permutomino is difficult to be described directly, its image through the transform F_n can be fully characterized. As a consequence, we were able to obtain an explicit formula for the number $\mathscr{C}_n$ of convex permutominoes (Corollary 2). We point out that a recursive generation technique (namely, the ECO method) has been independently proposed in [4], where an equivalent formula counting the number of convex permutominoes has been found.

We conclude remarking that the generating function of $\{\mathscr{C}_n\}_n$ is algebraic. Hence, it would be interesting to investigate if there is a natural unambiguous context-free language which is in bijection with the class of convex permutominoes.

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